STRATHCLYDE

HYSTERESIS IN THE FUNDAMENTALS OF MACROECONOMICS

BY

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Hysteresis in the fundamentals of macroeconomics

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Abstract

Two fundamental problems in economic analysis concern the determination of aggregate output, and the determination of market prices and quantities. The way economic adjustments are made at the micro level suggests that the history of shocks to the economic environment matters. This paper presents tractable approach for introducing hysteresis into models of how aggregate output and market prices and quantities are determined.

1 Introduction

"The roots of mainstream models in economics can be found in the neoclassical revolution of the 1870s. Neoclassical economists,such as Jevons, Walras and Fisher,used methods drawn from mathematical physics to analyse economic systems (Mirowski, 1989). The reformulations of economic theory on an axiomatic basis from the 1930s onwards retained key properties imported with the analytical methods, such as conservation and reversibility. In this world temporary shocks do not have permanent effects, and economic systems can retrace their steps when perturbed away from equilibria (see Colander et al., 2009) for a more general critique). Marshall was aware of the limitations of this method of analysis, arguing that "–if the normal production of a commodity increases and afterwards diminishes to its old amount, the demand price and the supply price are not likely to return, as the pure theory assumes that they will, to their old positions for that amount" (Marshall, 1890, p.426). At a more aggregate level, Keynes answered his question "is the economic system self-adjusting?" in the negative (Keynes, 1934). Hysteresis,involving temporary causes that can have permanent effects and non-reversible adjustment paths, has occasionally been postulated to be relevant to economic systems, but has not been formally incorporated into mainstream economic models. The present paper outlines methods for analysing economic systems haunted by hysteresis. Section 2 outlines mathematical methods for modelling hysteresis in the determination of aggregate output, Section 3 deals with market supply and demand, introducing hysteresis by way of the supply side of the market. The contribution of this paper can be seen as pointing out that there are tractable methods for modelling hysteresis in economic systems, and that these methods yield models that have some plausible and interesting properties."

1.1 Hysteresis

The term hysteresis was first coined in Ewing (1885) referring to "a persistence of previous states" observed when ferric materials are magnetised. The study of hysteresis produced a number of phenomenological and empirical models and techniques, of which the Preisach nonlinearity, introduced in Preisach (1935), is of greatest interest here. A mathematical framework and rigour was applied to models of hysteresis by a group of Russian mathematicians headed by M.A. Krasnosel'skii in the 1970s, (Krasnosel'skii and
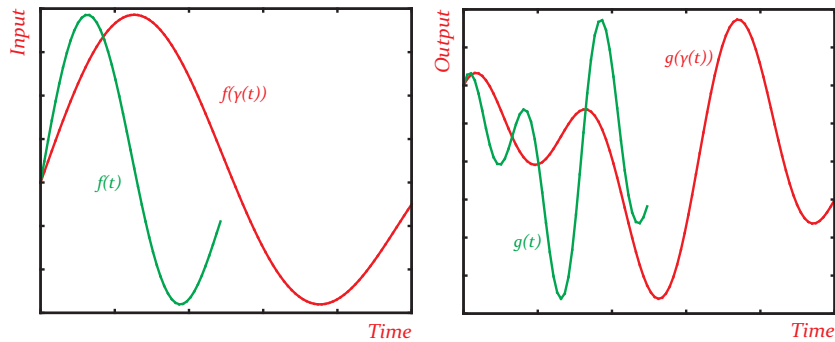
Figure 1: The action of a rate independent operator on an input function $f(t)$ produces the output function $g(t)$. Rate independence means that the action of the operator on the transformed input $f(\gamma(t))$, where $\gamma(t)$ is monotonically increasing, produces a similarly scaled output, $g(\gamma(t))$.

Pokrovskii, 1989). This general framework establishes hysteresis phenomena as input-output relationships. The general mathematical definition of hysteresis (omitting some technicalities) is that an operator which relates an input function to an output function is a *hysteresis operator* if the operator is both *deterministic* and *rate-independent* (see figure 1).

Complex hysteresis operators can be conceived as being constructed from simpler, elementary hysteresis operators, or *hysterons*. The exact type of hysteron used and the nature of the connection between them determine the properties of the complex operator which they combine to form. Once such an operator is constructed, its properties can be analysed and deduced from those of the simpler hysterons. Examples of these hysterons include the *Play* and *Stop* operators, and the operator which underlies the Preisach nonlinearity[1] the *non-ideal relay*[2]. For more details on the fundamentals of hysteresis models including the Preisach nonlinearity and their application, see Krasnosel'skii and Pokrovskii (1989), Mayergoyz (1991), Visintin (1994), Brokate and Sprekels (1996), Mayergoyz (2003) and Bertotti and Mayergoyz (2006). Particular applications to economic systems can be found in Cross (1993), Göcke (2002) and Cross et al. (2009).

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[1]The terms Preisach operator, Preisach model and Preisach nonlinearity all refer to the same mathematical object, and can be used interchangeably.

[2]Also termed the thermostat nonlinearity, lazy switch or hysteretic relay.

2 Macroeconomic flows

It was already mentioned in the introduction that the current mainstream models of macroeconomics originate in the "neoclassical revolution" period of the late nineteenth century. The protagonists in this revolution, such as Walras, Edgeworth and Jevons, applied paradigms and analogies drawn from Newtonian mechanics to economic systems.[3] A commonly used metaphor compared economic systems with a set of connected reservoirs at different levels, Edgeworth himself is one of those who used this analogy. Later economists such as Fisher (1925), and Phillips (1950), constructed actual hydro-mechanical machines for the determination of market prices and of macroeconomic flow variables such as output (respectively). Indeed, a number of "Phillips machines", or MONIACs, were constructed to order, both for study and policy making, see figure 2.

The neoclassical economic model was reformulated and axiomatised from the 1930s onwards, however many of the original analogies and paradigms were retained. In the modern version, decisions of production and consumption are undertaken by "representative" agents, who respond "smoothly" to variations in economic variables, often with linear responses. This is entirely in keeping with the "reservoir" model — the behaviour of a representative agent deciding between two states (say, an investor deciding whether to hold assets in US dollars or in Euro) can be modelled with two connected reservoirs. The volume of water in each determines the proportion of holdings in dollars and euro respectively, and the relative height difference between them represents the economic variables (in this case possibly the relative interest rates for each currency). This is illustrated in figure 3.

A flaw in the mainstream model, as illustrated by the reservoir analogy, is the assumption that economic agents behave homogenously, and that their collective behaviour can be captured by looking at a "representative" agent. The reality is somewhat different — for example, financial capital flows may be intended for start-up of production, or it may be intended as a deposit for a fixed term. Furthermore, different agents use different methods to predict

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[3]William Stanley Jevons studied chemistry, mathematics and logic at University College London, while Francis Ysidro Edgeworth (a close friend and neighbour of Jevons) taught himself mathematics and statistics, an interest perhaps influenced by William Rowan Hamilton, who was a friend of his father's. Leon Walras' father was also an economist, and encouraged his son to pursue the study of mathematical economics (see the biographies in Fonseca, web page).

Figure 2: Phillips' Economic Computer, the MONIAC (Monetary National Income Automatic Computer), as described in Phillips (1950). About 13 of these were built for various organisations worldwide, carefully calibrated to the economy of the destination country.
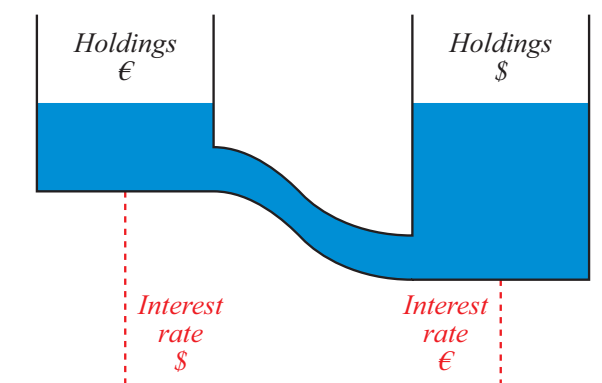
Figure 3: A representative investor modelled by two reservoirs. The investors funds will flow smoothly towards the currency where interest rates are higher. A reversal of a change in interest rates will reverse the corresponding change in holdings.

the future behaviour of the markets, and so two agents today could have quite different expectations of future returns. There is a further problem with this assumption from a mathematical point of view — the behaviour of the aggregate of a large number of individual agents may not be similar to the behaviour of the "average" agent. A good example of this is seen in the Preisach nonlinearity, where a "representative" agent would be a non-ideal relay (which is discontinuous), whereas the aggregate behaviour is given by the relatively smooth overall output of the Preisach nonlinearity.

A second difficulty lies in the implicit assumption that the "flows" can be reversed without cost. This assumption does not stand up to scrutiny. An increase of production will involve some costs which cannot be recovered if the increase is reversed; a change in equity or bond investments requires transaction costs; many deposits are fixed term and incur penalty charges if the investor does not complete the term; the act of making new decisions, of re-forming expectations, itself incurs a cognitive cost. These "sunk" costs give rise to infrequent, relatively large adjustments to economic behaviour, rather than almost continuous small corrections. The presence of substantial uncertainty in some situations further increases the tendency to "wait it out", and make large changes once the situation is more clear. This is clearly divergent from the analogy of fluid flowing between reservoirs.
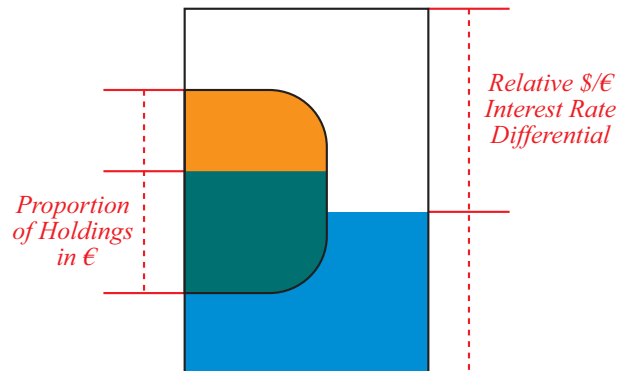
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Figure 4: A simple example of a porous flow model in an economic context. The proportion of stocks and funds held as € relative to $ is modelled as the water content of a sponge, with the exterior water level (or the water *potential*) describing the relative interest rate differential.

2.1 "Porous" flows

A better analogy may be the dynamics of the water content of a porous medium, for example a body of soil or a large sponge. On a small scale, the "sponge" is composed of pores, each of which can wither be empty or contain water. Due to surface tension there is a "cost" incurred in changing from one of these states to the other, in the "currency" of physics — energy. This cost leads to abrupt changes of the water content of a pore — the pore cannot be partially full as it is energetically unfavourable. To develop this analogy for our test case, we let the stock of financial assets held in euro (as opposed to dollars) be described by the water content of a sponge. The sponge is "attached" to the wall of a container, and the level of water in the container is used to describe the relative interest rate differential between euro and dollar deposits.

The question then is whether this new analogy is useful in some sense — does it give a more illuminating description of the behaviour of macroeconomic flows. This question can be addressed by modelling this type of system. New models to describe fluids in porous media have recently been proposed, using new types of equation. This paper will outline the economic rationale of a simple economic model which draws on the "sponge" analogy and discuss qualitative features of the model in the context of macroeconomic modelling.

2.2 Heterostasis and irreversibility

Mainstream macroeconomic models, derived from neoclassical theories, assume that equilibrim time paths are unaffected by actual economic outcomes. An example is in aggregate output flow — periods in which the actual output contracts (recessions) and periods of vigourous expansion (booms) in output are taken to have no lasting impact on the *equilibrium* growth rate. This feature of standard models, such as the "plucking" model of Friedman (1993), contrasts with alternatives such as in Hamilton (1989), in which (for example) recessions cause a permanent lowering of the output growth path.

The return to equilibrium in the "plucking" models arises from diminishing returns to capital in production. Thus if a negative (positive) shock reduces (increases) capital per effective worker below (above) its equilibrium level, the higher (lower) marginal return to capital would stimulate an increase (reduction) in investment in capital which would be strong but diminishing as capital, and hence the output produced with it, returns to the equilibrium value. In the alternative accounts, either production relationships involve constant returns per effective worker, or sunk costs introduce irreversibilities to production and capital stock adjustments. The term *heterostasis* is used to describe such phenomena — where the equilibrium value is permanently changed by a temporary stimulus.

Some recent empirical studies document recession curses. Cerra and Saxena (2005) used World Bank data for 192 countries from 1960 – 2001 to investigate whether recovery from recession following financial crises, wars and so on is associated with a return to the pre-recession trend value. The Calvo et al. (2006) study of the "Phoenix miracle" of recovery from recession deals with financial crises in emerging market economies 1980 – 2004, and with the recovery from the US Great Depression of 1929 – 1932. The key finding is that although recoveries are steep, output regaining its pre-crisis level within three years of the recession trough, the recoveries left the economies in question below the pre-crisis trend or equilibrium level of output. The implication is that countries experiencing frequent crisis-induced recessions, as in sub-Saharan Africa, tend to have low trend or equilibrium output growth rates. Of particular interest, in view of the financial crisis affecting much of the world since 2007, are the Cerra and Saxena (2008) estimates of the permanent effectts on output of the recessions following such crises. In high income countries, for example, the impulse response functions estimated indicate a 15% lower value for GDP ten years after the crisis.

The recent literature on boom blessings has largely concentrated on the extent to which the US asset market boom of the 1990s left lasting effects on productivity and equilibrium output growth in its wake. Caballero et al. (2006) provide an analysis of how the stock market boom could have led to a feedback from higher output growth to a lower long-term cost of capital. They paraphrase Keynes on the investment boom preceding the US Great Depression: "while some part of the investment which was going on was doubtless ill judged and unfruitful, there can, I think, be no doubt that the world was enormously enriched by the construction of the quinquennium from 1925 to 1929 . . . its wealth expanded in those five years by as much as in any other ten or twenty years in its history . . . a few more quinquennia of equal activity might, indeed, have brought us near to the economics Eldorado where all our reasonable economic needs would be satisfied" (Keynes 1931, cited in Keynes (1934, p.1178). Kindleberger (2000) provides an historical review).

2.3   A simple example

Consider a simple case where the relative price of capital in terms of output in normalised to unity, so that one unit of capital is used to produce one unit of output. Each firm, or more realistically, each operational unit within actual or potentially viable firms, has a choice between using its resources to be active, in the sense of using capital to produce output, or of being inactive and leaving any net resources on deposit in a bank. Each "firm" can thus be considered as an "elementary carrier of economic interests" (ECEI) that can switch back and forth between two different modes of behaviour: active and producing one unit of output, or inactive and producing zero units of output. The state of the aggregate macroeconomic environment, or the control variable, is a single number $I(t)$ which is the cost of borrowing faced by all "firms", being a markup over the repo rate set by the central bank. In this framework aggregate output is equal to the number of active "firms", $x(t)$. The dynamics of x are modelled by a continuous function that takes values within the unit interval $0 \leq x \leq 1$.

The aim is to describe the dynamics of $x(t)$ after some $t_0 = 0$. For convenience suppose that $x(0) = 1$. The control, or input, variable depends on the repo interest rate, i, set by the central bank. So we write the input variable as $I(t) = \{$Function of $i(t)\}$. The distinction is made between a value of the function $I(\tau)$ at a particular time moment $t \geq 0$, and the whole

prehistory $I_\tau(\cdot)$ of the variability of intensity I from the reference moment 0 up to the moment τ. For a given $\tau > 0$ the value $I(\tau)$ is a number, whereas $I_\tau(\cdot)$ is a function defined on $0 \leq t \leq \tau$. The next step is to suggest that, ignoring other factors, the value $x(\tau)$ at a certain moment $\tau > 0$ depends not on $I(\tau)$ but on the prehistory of the input variable, $I_\tau(\cdot)$. This implies the relationship $\dot{x}(\tau) = \{\text{Function of the prehistory } I_\tau(\cdot)\}$, i.e., the rate of change of the relative number of active firms is a function of $I_\tau(\cdot)$. Without the recession curse and boom blessing phenomena outlined earlier, the relationships between economic activity and interest rates might be captured by an ordinary differential equation such as $\dot{x}(t) = F(I(t), x(t))$. The question arises: *How can heterostatic phenomena be incorporated into a mathematical model of the macroeconomic dynamics of $x(t)$?*

2.3.1 Basic Assumptions

The basic assumptions are all based on taking a "wide-angle" view of the dynamics. In other words, the short-run volatility and variability is smoothed over by looking at the longer view. This allows the use of some very powerful mathematical tools — for example, it means the function $x(t)$ can be considered to be continuous as described above.

Assumption 1. *A restriction $I_\tau(\cdot)$ of the input $I(\cdot)$ to some interval $0 \leq t \leq \tau$ uniquely defines the corresponding restriction $x_\tau(\cdot)$.*

This basically makes the assumption that the dynamics of the system are deterministic — there is no uncertainty, and no randomness in the relationship between $I(\cdot)$ and $x(\cdot)$ (although this does not mean that the input variable $I(t)$ cannot contain a random element). It is emphasised that the *value* $I(\tau)$ does not necessarily determine the value $x(\tau)$, the entire prehistory $I_\tau(\cdot)$ up to the time τ may be needed to find the value $x(\tau)$ (indeed this will turn out to be the case).

In mathematical terms assumption 1 implies the existence of an operator, say W, that relates the input function $I_\tau(\cdot)$ to the output function $x_\tau(\cdot)$, i.e.

$$x_\tau(\cdot) = W I_\tau(\cdot).$$

From the discussion earlier it is expected that some memory-type effects might manifest themselves in the $I - x$ relationship, and these effects will be determined by the form which the operator W will finally take. Having said

10

that, the Volterra property described in Krasnosel'skii and Pokrovskii (1989) must hold: for any $0 < \sigma < \tau$ the function $x_\sigma(\cdot) = WI_\sigma(\cdot)$ must coincide with the restriction of the function $x_\tau(\cdot)$ to the interval $0 \le t \le \sigma$. In other words, the future behaviour of the input function cannot have any effect on the dynamics of x.

Assumption 2. *The number, $x(t)$, of "active" firms changes smoothly, and its rate of change with respect to time is denoted $\dot{x}(t)$.*

This assumption is reasonable provided the size of the system considered is large, as is the case for a macroeconomic system with a large number of firms.

Assumption 3. *At any given time moment τ there exists a unique "equilibrium rate", which is denoted $y = y(\tau)$.*

The equilibrium rate is defined as being that hypothetical value of the input variable which, if instantaneously achieved by the actual input variable, would cause the activity rate to remain constant. That is, if $I(t) = y(t)$ for all $t \ge \tau$, then the activity level remains constant: $x(t) \equiv x(\tau)$, $t \ge (\tau)$. It is emphasised that the behaviour in areas *separated from equilibrium* is of interest — it is not expected that $I(t)$ will remain at the equilibrium level. Further, the equilibrium rate $y(t)$ will likely also depend on the prehistory of the problem, an obvious example of this would be that the "equilibrium rate" at which activity would remain constant would be different if the current state was reached by a large recession than if the same state was reached by a boom period.

Assumption 4. *The rate of change of the level of activity at a time t is proportional to the difference between the actual input variable $I(t)$ and the equilibrium rate $y(t)$.*

$$\dot{x}(t) = k\big(I(t) - y(t)\big). \tag{2.1}$$

This relationship is analogous to Darcy's law for porous flows, with I the analogue of the external potential, y of the matric potential and x the water content of the porous medium (with k a conductivity parameter). It can also be seen as analogous to Ohm's law in electric circuits: $I - y$ is a potential difference, x is a current and k is electrical conductivity. In physical situations the "potentials" have units of energy — the "cost" of work. The

natural correspondence is that the rates I and y are in "cost of activity" units (actually, in our context, "cost of capital").

The differential relationship (2.1) is not yet "closed" — it cannot be solved numerically, analytically, or produce results which could verify or falsify the assumptions made. In order to close the system, some relationship between the "equilibrium rate" $y(t)$ and the current level of activity $x(t)$ needs to be established.

2.3.2 Closing the equation

In order to complete the relationship between $x(\cdot)$ and $I(\cdot)$, the differential equation (2.1) needs to be closed by establishing a relationship between $x(\cdot)$ and $y(\cdot)$. This relationship only admits certain pairs of functions $y(\cdot)$ and $x(\cdot)$. The totality of such pairs, $(x(\cdot), y(\cdot))$ which are possible in our system is denoted Π, where both components $x(\cdot)$ and $y(\cdot)$ are defined for the same interval $0 \leq t \leq \tau_0$. For a *particular* interval τ, the subset of Π consisting only of pairs defined for $0 \leq t \leq \tau$ is denoted Π_τ.

Assumption 5. *The totality Π is rate-independent: i.e. if a pair $(x(\cdot), y(\cdot))$, $0 \leq t \leq \tau$, is in Π_τ, then for any positive γ the pair $(x_\gamma(\cdot), y_\gamma(\cdot))$ given by $(x_\gamma(t), y_\gamma(t)) = (x(\gamma t), y(\gamma t))$, $0 \leq t \leq \tau/\gamma$, belongs to $\Pi_{\tau/\gamma}$. This is the technical definition of rate-independence as introduced in section 1.1.*

This means that a scaling of the rate of change of the input results in the same scaling being applied to the output, and is shown in figure 1. Note that suggesting rate independence of $(x(\cdot), I(\cdot))$ or $(y(\cdot), I(\cdot))$ pairs would not be appropriate, as transient processes would be ignored. This directly corresponds to recent developments in soil hydrology, where the moisture content and matric potential are related in a rate independent manner. Thus this is the first appearance of the new "sponge" analogy in our model.

Assumption 6. *For any given function $y(t)$, $0 \leq t \leq \tau$, there exists exactly one function $x(t)$, $0 \leq t \leq \tau$ such that $(x(\cdot), y(\cdot)) \in \Pi$.*

This is equivalent to the hypothesis that for any admissible function $y(t)$, $0 \leq t \leq \tau$, the corresponding value $x(\tau)$ is uniquely defined. However it is emphasized that this value $x(\tau)$ cannot be uniquely defined in terms of the number $y(\tau)$ only. Taking into account assumption 5, it is clear that all relevant information about the prehistory $y_\tau(\cdot)$ may be condensed into the form of a sequence $SV(y_\tau(\cdot))$ of the shock values, i.e. into the sequence of the

alternating locally maximal and locally minimal values of the function $y_\tau(\cdot)$. An example of the special role of the shock values is the awareness of technical traders of *supports* and *resistances* — shock values which a commodity does not easily break through.

In mathematical terms, this means that there must exist an operator G which relates each unique function $x(\cdot)$ with its pair in Π, i.e. $x(\cdot) = Gy(\cdot)$, such that for any $t \in [0, \tau]$, the value $y(t)$ is the equilibrium rate for the function $x(\cdot)$ at the moment t. From assumption 5, it is clear that this operator G must also be rate-independent. The general relationship (2.1) can now be written

$$\dot{x}(t) = k\big(I(t) - y(t)\big),$$
$$x(\cdot) = Gy(\cdot)$$

All that remains in order to close the system is a useful and justifiable form for the rate independent operator G. The theory of rate independent memory operators has been developing rapidly in recent years. As described in section 1.1, such operators are called *hysteresis* operators. The reader is referred again to the fundamental texts (Krasnosel'skii and Pokrovskii (1989), Mayergoyz (1991), Brokate and Sprekels (1996), Visintin (1994)) and the recent three-volume set (Bertotti and Mayergoyz, 2006) surveying the current state of research in hysteresis.

2.3.3 The operator G

In order to justify a particular form for the operator G, the "slow-time" limit of the process is examined. Given a function $y(t)$, $0 \le t \le \tau$, and $x(\cdot) = Gy(\cdot)$, consider a hypothetical "slow" function

$$y_\gamma(t) = y(\gamma t), \qquad\qquad 0 \le t \le \tau/\gamma$$

for small $\gamma \ll 1$. This function varies very "slowly" with respect to time, so $|\dot{y}_\gamma| \ll 1$. By assumption 6 there exists a well-defined counterpart $x_\gamma(\cdot) = Gy_\gamma(\cdot)$, and moreover by assumption 5 this function $x_\gamma(t) = x(\gamma t)$. There thus exists an input function $I_\gamma(t)$ such that

$$\dot{x}_\gamma(t) = k\big(I_\gamma(t) - y_\gamma(t)\big), \tag{2.2}$$

$$x_\gamma(\cdot) = Gy_\gamma(\cdot). \tag{2.3}$$

Since everything is "slow", $\dot{x}_\gamma$ is very small, and thus for sufficiently small γ:

$$I_\gamma(t) \approx y_\gamma(t). \tag{2.4}$$

Following from assumption 1, the function $x_\gamma(\cdot)$ can be understood as

$$x_\gamma(\cdot) = WI_\gamma(\cdot) \approx GI_\gamma(\cdot) \quad \text{since } I_\gamma(t) \approx y_\gamma(t). \tag{2.5}$$

For the remainder of the derivation the focus will be on describing this operator.

Assumption 7. *We treat the totality of agents in the economy as an infinite ensemble Ω, and we assume that members of Ω behave independently.*

The point in swapping a large finite set with an infinite set is purely technical: it is easier to integrate a continuous function, rather than to sum a finite series.

Assumption 8. *To each element $\omega \in \Omega$ there correspond two numbers, $\alpha(\omega)$ and $\beta(\omega)$, satisfying the inequalities $0 \leq \alpha(\omega) < \beta(\omega) \leq 1$. It is assumed that for the values $I(t) \geq \beta(\omega)$ the only equilibrium mode of behavior of the agent ω is to be active in the economy, and for values $I(t) \leq \alpha(\omega)$ the only equilibrium behavior is to be inactive. For $\alpha(\omega) < I(t) < \beta(\omega)$ the agent ω has two possible modes of equilibrium behaviors: activity or inactivity, depending on its prehistory. Finally, the agent ω is "lazy": it does not change behaviour as long as the input $I(t)$ varies within the boundaries $\alpha(\omega), \beta(\omega)$.*

This assumption captures two features of economic systems which are of interest here. First, it does not assume homogeneity of economic agents — rather than using a representative agent an entire ensemble of different agents is considered. Second, changes in behaviour of any one agent are made discontinuously and are not easily reversed. Both of these behaviours were discussed earlier in this section, and assumptions 7 and 8 incorporate them into the model system.

To formalise this behaviour the nonlinear operator $R_{\alpha,\beta}$ is introduced for given numbers $0 < \alpha < \beta < 1$. This operator is termed the non-ideal relay, and here describes the behaviour of an individual agent, ω. The variable output $z(t) = R_{\alpha,\beta}[t_0, \eta_0]\, y(t)$, $t \geq t_0$, depends on both the variable input $y(t)$, $t \geq t_0$, which is an arbitrary continuous function, and the initial state η_0, which is either 0 or 1. The resulting function $z(t)$ has at most a finite number of discontinuities on any finite interval. The non-ideal relay (also known as a "homeostatic nonlinearity") is a common and important building block in many subject areas, and is fundamental to the study of hysteresis — see, for example, Krasnosel'skii and Pokrovskii (1989). Below for $t_0 = 0$, $\eta_0 = 0$ the notation $R_{\alpha,\beta}y(t)$ is used to mean $R_{\alpha,\beta}[t_0, \eta_0]\, y(t)$.

Assumption 9. *The pairs, $(\alpha(\omega), \beta(\omega))$, $\omega \in \Omega$, of thresholds are distributed with some integrable density $\mu(\alpha, \beta)$.*

It is now possible to write

$$(WI_\gamma)(t) = \int_0^1 \int_\alpha^1 z(\alpha, \beta)\mu(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta$$
$$z(\alpha, \beta) = \big(R_{\alpha,\beta}I_\gamma\big)(t), \qquad\qquad t \geq 0$$

This is an expression of the Preisach nonlinearity, which was introduced in the context of ferromagnetism by Preisach (1935), but has found a more general applicability. In hydrology the *independent domain* model is equivalent to the Preisach nonlinearity, and was developed in parallel to it by Néel (1942,1943), Everett and Whitton (1952) and others. For a succinct explanation of the development of the Preisach model, and its relation to the independent domain model, see Mayergoyz (2003).

Returning to statements (2.3),(2.5), the relations become

$$(Gy_\gamma)(t) = \int_0^1 \int_\alpha^1 z(\alpha, \beta)\mu(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta \tag{2.6}$$
$$z(\alpha, \beta) = \big(R_{\alpha,\beta}y_\gamma\big)(t).$$

Thus the principle system of equations can be written

$$\dot{x}(t) = k\big(I(t) - y(t)\big)$$
$$x(t) = \int_0^1 \int_\alpha^1 z(\alpha, \beta)\mu(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta \tag{2.7}$$
$$z(\alpha, \beta) = \big(R_{\alpha,\beta}y_\gamma\big)(t).$$

Once a density function $\mu(\alpha, \beta)$ (sometimes called a Preisach function) is given this is a closed differential-operator system of equations. What form of density is suitable for use in this economic context is an open question. In this section a so-called "wedge density", as described in Flynn et al. (2006) and McNamara (2008), is used for demonstration purposes. The use of this class of density follows from the analogy with fluid in a porous medium (wedge densities were first used in describing soil-water hysteresis), but not from any compelling empirical justification. A shorter notation for (2.7) writes $\mathcal{P}[\eta_0]y$ for the Preisach nonlinearity with a given density and an initial state η_0. Then the system of equations can be expressed as

$$
\begin{aligned}
\dot{x}(t) &= k\big(I(t) - y(t)\big) \\
x(t) &= \big(\mathcal{P}[\eta_0]y\big)(t)
\end{aligned}
\tag{2.8}
$$

This is a new type of equation, which has only recently been studied. The key feature of the equation, which is more clearly seen in the compact notation, is that the action of the Preisach nonlinearity is *under* the highest derivative in the equations. This contrasts with the relatively well studied case of the Preisach nonlinearity on the right-hand side of such differential operator equations (i.e. $\dot{x} = f(x, t) + (\mathcal{P}x)(t)$). Some of the main questions in a mathematical sense, such as existence and uniqueness of solutions, have been addressed, see Flynn and Rasskazov (2005) and the references therein. What remains in this application is to examine the behaviour of this system in a qualitative sense, and assess whether it could be useful in an economic context. The next section attempts to address this question.

2.4   Dynamics of the system

In this section some of the main qualitative features of the simple model derived above are explored, with particular interest in the relevance to modelling macroeconomic flows. The algorithm used for producing numerical trajectories of the system in (2.8) is taken from Flynn and Rasskazov (2005), and implemented in C++ (see appendix in McNamara (2008) for further details).

Looking at (2.8):

$$
\begin{aligned}
\dot{x}(t) &= k\big(I(t) - y(t)\big) \\
x(t) &= \big(\mathcal{P}[\eta_0]y\big)(t)
\end{aligned}
$$

there are three different components of the system — $I(t)$ which is the "input" or control rate; $y(t)$ which is the "equilibrium" rate; and $x(t)$, which is the activity level (the rate of flow). In control terms, and since $x(\cdot)$ and $y(\cdot)$ are "simply" related by the Preisach nonlinearity, the two functions $x(\cdot)$ and $y(\cdot)$ are "outputs" of the system, while $I(\cdot)$ is the sole input. In this presentation, the main qualitative features of the model are being investigated, so the exact form of the density function in (2.7) is not important.

An example trajectory of the system is illustrated in figure 5. There are a number of key features to be noted.

- The two "outputs" of the system – $x(t)$ and $y(t)$ – change direction at the same moments in time. In other words if $y(t_0)$ is an extremum of $y(t)$ then $x(t_0)$ is an extremum of $x(t)$.

- The graph of $y(t)$ changes direction whenever it crosses the graph of $I(t)$. This is expected from the form of the equation, where the right hand side involves $I(t) - y(t)$. This also means that turning points of y (and thus x) are somewhat delayed with respect to turning points of I.

- Turning points of $y(t)$ are "non-smooth", giving a "shark-toothed" appearance. I immediately following a turning point, $y(t)$ behaves in a near-linear fashion, and closely follows the graph of $I(t)$.

- Some evidence of heterostasis in the behaviour of $x(t)$ can be seen. I n particular, there is a clear "upward" trend in x from the left of the diagram to the right.

The asymmetry of $y(t)$ is of particular interest, as similar behaviour is seen in the dynamics of many economic indicators and financial stocks on intermediate to long timescales. Immediately following a turning point in $y(t)$ the rate of change of y is large in magnitude. Similar qualitative behaviour can be observed in real systems. An example is the asymmetry around turning points in the Dow Jones industrial share price Index. This is illustrated in figure 6.

2.4.1 Periodic inputs

An interesting question to examine is the behaviour of the system in response to periodic inputs of different frequencies. This can give some very useful
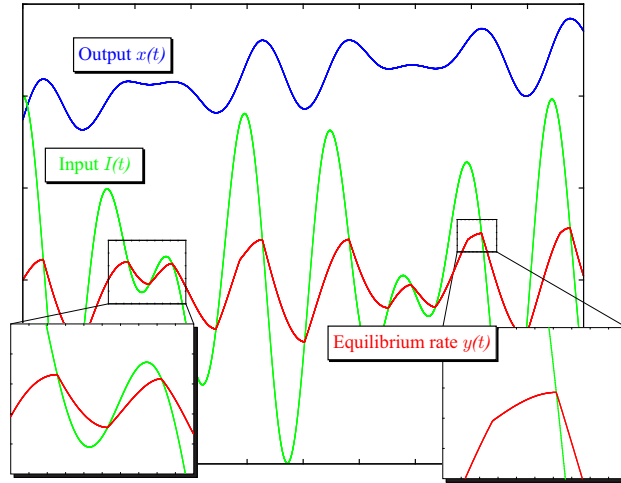
Figure 5: An example trajectory of the model equations, (2.8). The output $x(t)$ is on a different scale to the input, $I(t)$ and equilibrium rate, $y(t)$.
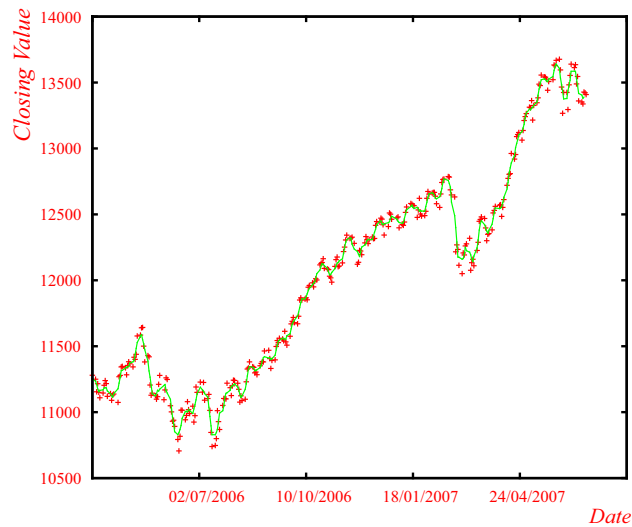

Figure 6: Historical data for the Dow Jones Industrial Average (DJIA). The turning points in July '06 and around February '07 illustrate an asymmetric behaviour similar to that of $y(t)$ in figure 5.

18

qualitative information, and is particularly interesting in the context of economics, where cycles of periods from less than a year to over 50 years have been identified, for example in Schumpeter (1939). Outputs corresponding to specific periodic inputs do not accumulate to give outputs of other inputs since the system is strongly nonlinear. However, the behavior of such simple outputs can still shed some light on the contributions to the output of the components of a general type of input.

Given a periodic input function, $I(t + T) = I(t)$, the corresponding outputs, $x(t)$ and $y(t)$, also become periodic, with the same period — after some transients. By plotting pairs of the three components of the system, a loop structure can be seen. Each such pair illustrates qualitative features of the system.

When two periodic functions are related by a Preisach nonlinearity and plotted against each other, the area enclosed by the loops is identified with dissipation caused by the action of the hysteresis. In physical systems these losses are easily identified, for example as energy lost to heat. In an economic context these losses are more difficult to pin down, but are likely to be caused by losses of potential output.

- *Behaviour of loops: y against I*

  Plotting the equilibrium rate, $y(t)$ against the input, $I(t)$, there are two behaviours which are of interest — the transient behaviour, which depends on initial conditions, and the convergent loops.

  The transient behaviour is shown in figure 7. The transient behaviour quickly dies down, and the trajectory converges to a closed loop. This loop is not dependent on the initial state of the system at $t = 0$, and so is entirely defined by the input function $I(t)$.

  The behaviour of the $I - y$ loops for different frequencies is shown in figure 8, where the transient is discarded to leave the converged loops for the different frequencies. These loops are vaguely elliptical in shape, with "corners" at the highest and lowest points. For each frequency, these corners lie on the same straight line, which implies a linear relationship between the frequency of $I(t)$ and the amplitude of $y(t)$. The equilibrium rate clearly responds more strongly to a slowly changing $I(t)$, approaching a linear relationship for very low frequencies. Very high frequency inputs, in contrast, result in an equilibrium rate which changes less and less, becoming almost constant. Note also that the loops have a common "centre" around which they are symmetrical.
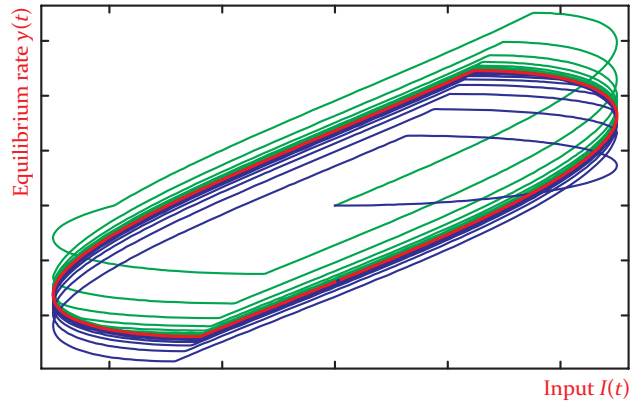
19

Figure 7: The transient behaviour of $y(t)$ for a simple periodic input function $I(t)$ with two different initial states. The two trajectories are in blue and green, it is clear that both converge to the same closed loop, shown in red.
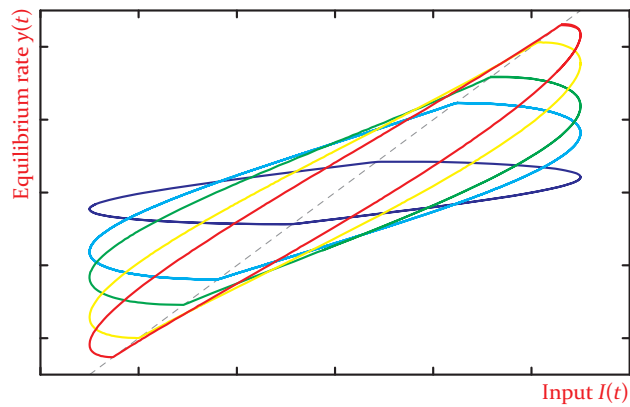

Figure 8: The long-time behaviour of $y(t)$ plotted against $I(t)$ for periodic inputs with different frequencies. The highest frequency in this image is shown in black, the lowest in red. The dashed black line passes through the corners on each loop.
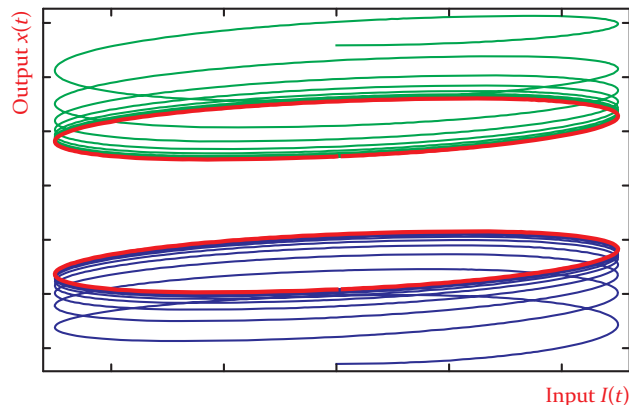
20

Figure 9: The transient behaviour of $x(t)$ for a given periodic $I(t)$, with two different initial states. The two trajectories are in blue and green, each converges to a different loop (both in red).

- *Behaviour of loops: x against I*
  Again, when plotting the output, $x(t)$, against the input, $I(t)$, both the transient and long-time behaviour are of interest. The transient behaviour is shown in figure 9. In this case the transient behaviour for different initial states leads to different outcomes. The shape of the loop to which each converges is the same, however they are displaced from each other by a constant.

  The $I - x$ loops for different frequencies are shown in figure 10. Very similar behaviour to that in figure 8 is seen, although the "corners" are not present here. Also different is the lack of a common "centre" around which the loops for all frequencies lie. Loops for lower frequencies have a higher *average* output, as well as a larger amplitude.

- *Behaviour of loops: x against y* The transient behaviour of $x - y$ loops is not as interesting as for the other cases. The final behaviour is very interesting however, and is shown in figure 11. Note that x and y are related by the Preisach nonlinearity, as stated in (2.8), i.e.:

$$x(t) = \big(\mathcal{P}[\eta_0]y\big)(t).$$

In other applications there is a direct correspondence between the area of these loops and the energy dissipated due to hysteresis. As men-
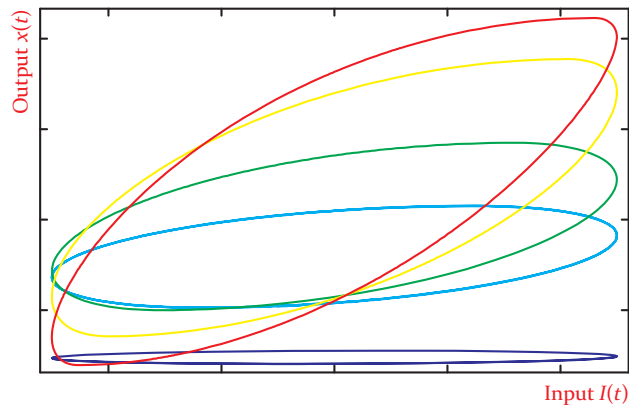
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Figure 10: The long-time behaviour of $x(t)$ for periodic inputs of different frequency. The initial state is the same in each case. The highest frequency is in black, the lowest in red.

tioned before, this can be interpreted as lost potential output in an economic context.

The areas of these loops were estimated for a large number of different periods. The results of this calculation as shown in figure 12. The "losses", per unit time, for higher frequency inputs are substantially lower than those for slower inputs.

2.4.2   The effect of temporary shocks

Of key importance to the model under consideration is the response to temporary stimuli — or shocks. As discussed in section 2.2, economic behaviour does not "forget" the effects of boom periods or recessions. This heterostasis, therefore, is an important feature for any macroeconomic model to display.

It has already been established that the long-term behaviour of the output function $x(t)$ depends specifically on the initial conditions of the system, while the equilibrium rate, $y(t)$, does not. This suggests that heterostasis is present in the $I - x$ relationship, but not in that of $I - y$.

To test this, a simple periodic input function $I(t)$ with a small shock was used as the input of the system. This function is plotted with the corresponding $y(t)$ and $x(t)$ in figure 13. As expected, $y(t)$ is disturbed by the shock, but quickly retains its former levels. In contrast $x(t)$, the output itself, is permanently altered by the influence of the shock, and does not
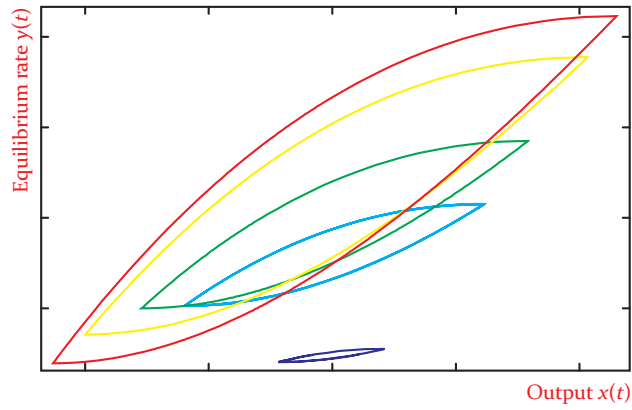
Figure 11: Graphs of the output, $x(t)$ against the equilibrium rate, $y(t)$ for different frequencies. The same colouring scheme is used as before.
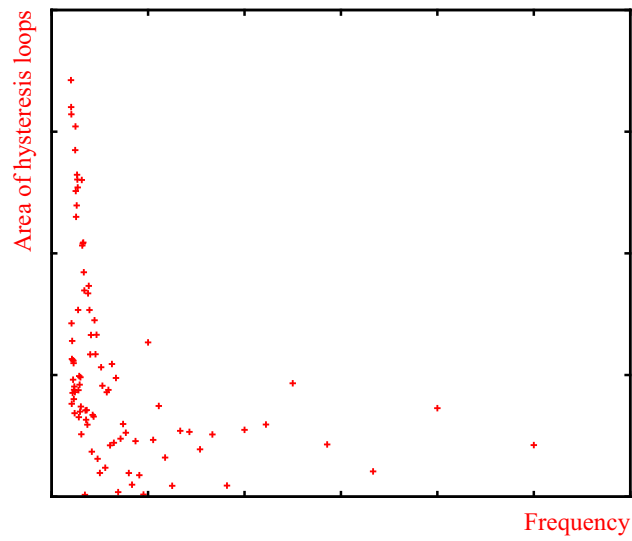

Figure 12: Estimates of the areas of $x - y$ loops per unit time, i.e. $\frac{1}{T} \int_0^T y(t) \mathrm{d}x(t)$.
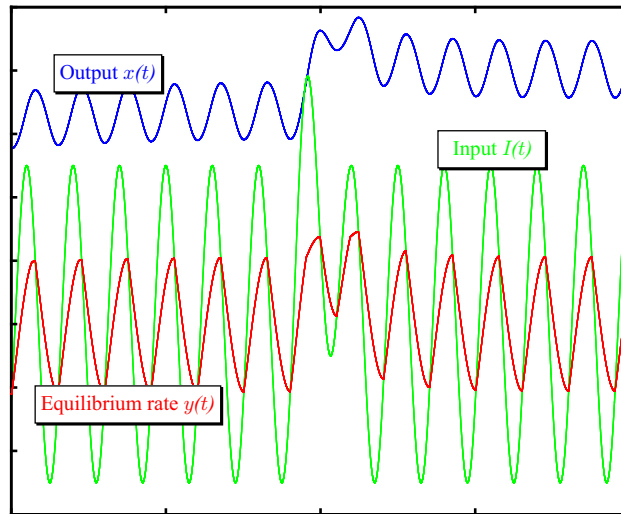
Figure 13: The effect of a temporary shock on the system. $I(t)$ is periodic except for a small region where a continuous perturbation is added. The resulting $y(t)$ quickly returns to pre-shock levels, $x(t)$ displays heterostatic behaviour.

return to previous levels.

This system is a particularly simple model, and can only serve as a prototype model in order to demonstrate the potential for models incorporating hysteresis. As such its clear derivation from first principles allows for qualitative behaviour to be matched closely to the associated assumptions, and the richness of the behaviour to be demonstrated. In order to be useful in a quantitative sense, however, work needs to be done in several directions, including the identification of a suitable Preisach density functions, and fitting to actual data. We return to this question after the next section.
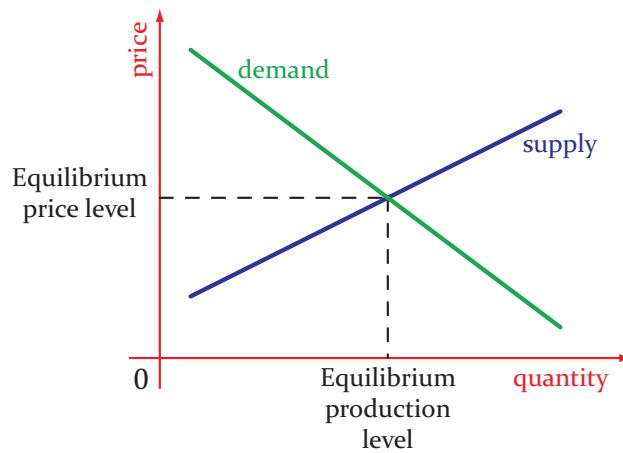
Figure 14: Supply and demand curves.

3 Hysteresis and price plasticity in analysis of supply and demand

In this section we apply the ideas developed in the previous section to another key aspect of fundamental macroeconomic thought. The analysis of supply and demand curves is one of the central ideas behind neoclassical qualitative economic analysis (Mankiw, 2006). They express the relationships between the prices of products or services and the aggregate quantities demanded by the consumers or potentially supplied by producers at the prices in question.

The standard illustration of the supply and demand curves is shown in figure 14. Often they are represented as segments of straight lines (approximating more complex nonlinear functions on finite intervals). Economics textbooks use the horizontal axis for quantity of the product and the vertical axis for the price. Generally, it is assumed that the supply curve $S = f_S(P)$, where S is the quantity that producers choose to supply to the market at a particular price P, is a monotonously increasing function of price P, and the demand curve $D = f_D(P)$, where D is the quantity demanded by consumers at a particular price P, is a monotonously decreasing function of price P.

The intersection of the two curves defines an equilibrium price and an equilibrium quantity of the product in question. The equilibrium price P_{eq}

is the solution of the equation:

$$f_S(\cdot) = f_D(\cdot),$$

and the equilibrium quantity Q_{eq} is defined by

$$Q_{eq} = S = D.$$

The classical assumptions are that (a) all the of processes in the economy occur at a slow pace without "frictions" (adiabatic process assumption); (b) the supply and demand curves are the same for rising and falling prices (symmetry assumption); (d) tastes and production technologies do not change when prices are away from their equilibrium values; and (d) these curves do not change on the characteristic time scale of interest. Under these assumptions the quantity of product on the market and the price of the product both reach their respective equilibrium values.

3.1 Supply with memory

In real markets not all of the assumptions of the classical supply–demand model are satisfied. In particular, the characteristic times of the various processes that take place in the economy do matter, and the final *equilibrium* production and price for a product or service may depend on these characteristic times. Marshall (1890) used the example of the burning of cotton fields during the US Civil War of the 1860s to point out that learning by consuming, or by producing, produced shifts in demand and supply that would remain after the cotton field were replanted after the war. In the often substantial time interval between investing into productive capacity and bringing a product to the market, the market price could well change. This does not mean, though, that the production originally planned given a particular expected product price will cease if the actual price turns out to be lower than expected. Once production has taken place on the basis of an expected product price $P_1 = \beta$ it will only be economically sensible to cease production if the actual market price falls to some $P_2 = \alpha < \beta$. This is due to the investment which has been *sunk* in the production process, which includes the costs of product-specific investment, the cost of expanding the workforce, etc., and is illustrated in figure 15. The values of β (start price) and α (stop price) are likely to be different for various companies, depending on the local
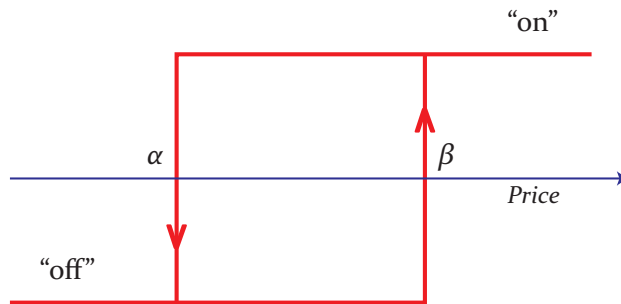
Figure 15: Individual firm's production decision. If price rises above β, production starts if it was not already in progress. If the price is below α, no production takes place. In the region between the two thresholds, either situation is possible. This type of system is known as a non-ideal relay, and underlies the Preisach nonlinearity

investment profile, the product-specificity of the investment, marketing exposure and other company-specific conditions. This means that the supply curve, in general, should behave differently for increases and decreases in price. This behaviour is related to the traditional notion of price elasticity in the same manner as plasticity is related to elasticity in mechanics, hence the term "price plasticity".

In the model that follows we retain a simple specification of the demand side of the market. Over the time frame used the shape of the demand curve is assumed to be fixed. A "par" price will be defined as one lying on this demand curve. In the illustrative example discussed below we use a straight line with a negative slope to describe demand curve behavior. In general, however, the demand curve could be a monotonically decreasing nonlinear function.

The main innovation comes in the specification of the supply side of the market (see Piscitelli et al., 1999). As the market price increases, new producers may choose to join the already active ones in production, or active producers may choose to expand their production. The increase from some initial price to a new higher price stimulates additional production and an increase in market supply. Thus the supply curve depends on the distribution of the thresholds at which firms switch from inactivity to activity (β) or in the other direction (α). This distribution of thresholds is a measure on the half plane $\beta > \alpha$, and in the continuum limit can be modelled by a density
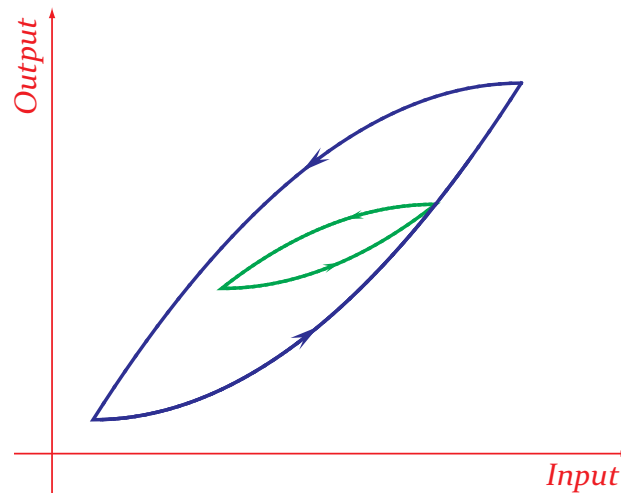
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Figure 16: The Preisach nonlinearity, illustrating the major hysteresis loop (blue) and a smaller secondary loop (green).

function (the *Preisach function* mentioned in the previous section) $\mu(\alpha, \beta)$. The Preisach nonlinearity (2.6) as described in the previous section describes the behaviour of such an aggregation of non-ideal relays. A good interactive demostration of the Preisach nonlinearity can be found at the interactive website by Flynn et al. (web page). A major effect of Preisach hysteretic behaviour is that the supply relationship now also depends upon the prehistory of the price, and not just the current value of the price. The "loop" behaviour of hysteretic systems, in particular the Preisach nonlinearity, is shown in figure 16. For a given value of the input (price in our application), an interval of possible outputs (supply here) exists. The particular branch taken is determined by the prehistory of the input — in particular the *non-dominated* extremum values of the input.

There is a finite adjustment time for both prices and production. If the system finds itself shocked away from a particular equilibrium it will try to restore some other equilibrium through price adjustments and/or changes in the level of production. However, these changes are not instantaneous. The characteristic times of price and output adjustments (or, more exactly, their inverses) are parameters in the model.

3.2 Model definition

The dynamics of the system described above can be presented in terms of the variables Q (total quantity of production) and P (market price) by the following system of operator-differential equations incorporating Preisach hysteresis.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k_1 \left(S - Q \right),$$
$$\frac{\mathrm{d}P}{\mathrm{d}t} = k_2 \left(P_{par} - P \right),$$
$$S = \mathcal{P} \left[\eta_0 \right] P, \qquad \text{(hysteresis link)}$$
$$P_{par} = F_D(Q).$$

The parameters k_1 and k_2 are, as mentioned, the inverse characteristic adjustment times for production and price respectively. The interaction links are $S = \mathcal{P} \left[\eta_0 \right] P$, which is a Preisach nonlinearity, the main innovation in this model; and $P_{par} = F_D(Q)$, which is the "par" price based upon the demand curve. In other words, $F_D(Q)$ is the price that would be requred to clear the market given a production quantity of Q. The demand curve is assumed to be fixed over the time-scale of the model.

The system is determined fully by the choice of suitable initial conditions. These include the initial price and production levels, P_0 and Q_0 respectively, and an initial state, η_0, for the Preisach nonlinearity in the hysteresis link. The system will evolve to an equilibrium state, as in the classical model. In this model, however, a range of possible equilibrium price–output pairs arises. This is illustrated in figure 17, where the envelope curves of the hysteresis link are plotted with the demand curve. The particular demand curve and Preisach nonlinearity shown in the figures were chosen for illustrative purposes.[4]

This system is closed by the choice of initial conditions — both the initial price-output pair, (P_0, Q_0), *and* an initial state for the Preisach nonlinearity, which encodes prior variations in the price. The eventual range of equilibrium values of P and Q are determined by these choices. An extreme example is

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[4]The Preisach nonlinearity in these illustration uses a "uniform" density, where all threshold pairs (α, β) are equally likely. This is purely for simplicity because the type of density functions to use in economic models is an open question. In an econometric application to explaining unemployment time series (Darby et al., 2006) the estimates were not sensitive to the density functions used.
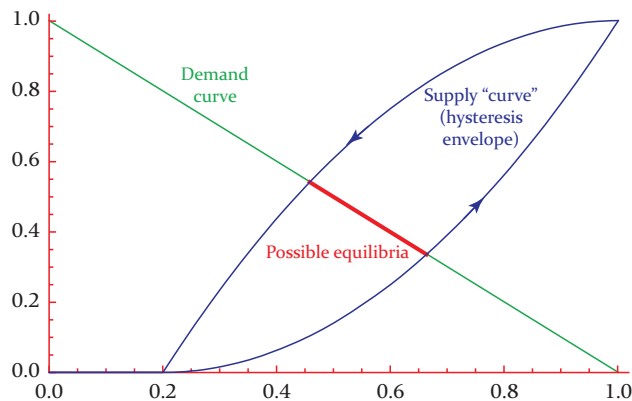
Figure 17: Hysteresis envelope (blue) and demand curve (green), showing the range of allowable equilibrium points (red). Any point within the envelope is an allowable input-output pair for the Preisach nonlinearity. This leads to an interval of allowable equilibrium prices. The particular price "chosen" by the system depends on the prehistory of price variations.

given in figure 18, where the same initial price and output conditions give rise to very different equilibria due to differences in the initial state of the Preisach nonlinearity. This extreme example brings out the qualitative property that the equilibrium price and output levels depend on past variations in the market price. Introducing hysteretic effects into a model of supply and demand dynamics provides for a much richer behaviour, where past shocks have permanent effect on market outcomes.

4   Conclusions

The purpose of this paper has been to investigate the implications of relaxing some fundamental assumptions in mainstream macroeconomic theory. When applied to some very simple example models, this change leads to heterostasis and the persistence of the effects of temporary shocks. Hysteresis effects are very plausible characteristics of economic behaviour, and the simple models outlined show how hysteresis can be introduced into the bedrock of economic assumptions. Further, the use of the Preisach nonlinearity is a link between micro-level behaviour, in the form of the non-ideal relay, and aggregate macroeconomic outcomes.
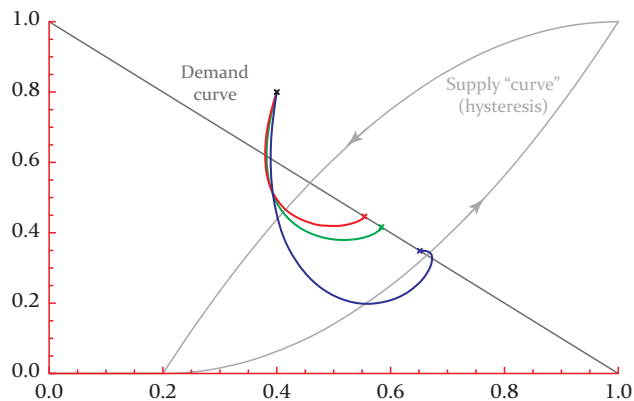
Figure 18: Evolution of the system to equilibrium price/production levels. The same initial values P_0 and Q_0 give rise to different equilibria due to the influence of historical price variations.

A major open question in efforts to use the Preisach nonlinearity in macroeconomic modelling is the identification problem — finding a suitable Preisach density function for the distribution decision thresholds. Empirical implementation requires being able to identify such density functions. Some of the suitable data is not publicly available due to commercial sensitivity (an example considered in Twomey (2008) is the Irish mobile phone market). A further possibility is the use of experimental methods to identify switching points and their distributions.

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