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CONTESTS WITH GENERAL PREFERENCES

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Abstract

This article investigates contests when heterogeneous players compete to obtain a share of a prize. We prove the existence and uniqueness of the Nash equilibrium when players have general preference structures. Our results show that many of the standard conclusions obtained in the analysis of contests—such as aggregate effort increasing in the size of the prize and the dissipation ratio invariant to the size of the prize—may no longer hold under a general preference setting. We derive the key conditions on preferences, which involve the rate of change of the marginal rate of substitution between a player's share of the prize and their effort within the contest, under which these counter-intuitive results may hold. Our approach is able to nest conventional contest analysis—the study of (quasi-)linear preferences—as well as allowing for a much broader class of utility functions, which include both separable and non-separable utility structures.

Keywords: contest; general preferences; aggregative game.

JEL Classification: C72, D72

1 Introduction

Contests can be characterized by players expending sunk effort in order to obtain a prize. As this is a rather general economic phenomenon, contest theory has become a dominant framework to explain the incentive structures within a variety of applications, including—but not limited to—rent seeking, patent races, litigation, and conflict (Konrad, 2009). There are two key types of contest studied in the literature: those in which an indivisible prize is contested; and those in which a perfectly divisible prize is shared among contestants. With non-linear payoffs the two are not equivalent, and while the former has been well-studied, the latter has seen much less attention. Although there is large applicability of contests in which a perfectly divisible prize is shared among contestants, there remains a convention within the analysis of such contests to assume players have (quasi-)linear preferences. With such a variety of potential applications at hand, it is, however, perfectly plausible to envisage players with alternative preference structures. Thus the question arises as to whether a tractable analysis of contests with more general preference structures can be undertaken.

In this article we provide a framework where players with general preference (utility) structures contest a share of a prize. We show both the existence and uniqueness of a Nash equilibrium and characterize players' equilibrium effort levels. We assume conventional—but general—restrictions on players' utility functions (such as utility increasing in prize share and decreasing in effort) and can therefore analyze a broad class of potential contest applications. Within such a setting, we show a fundamental component is the rate of change of the marginal rate of substitution between players' prize share and their contest effort as the size of their share of the prize changes. In particular, we find that this, which is always monotonically decreasing in the standard contest model, has dramatic effects on how aggregate contest efforts (and the ratio of prize dissipation) change with respect to the size of the prize.

Within the classical literature on contests (e.g., Congleton et al., 2008), a standard result exists in which aggregate efforts are increasing in the size of the prize. Furthermore, it is also usually observed that the dissipation ratio of the prize is independent of the prize but increasing in the number of players. Thus as the number of players increases, the dissipation ratio tends to one (e.g, Hillman and Samet, 1987). Yet both these predictions—of increasing aggregate effort and invariance of the dissipation ratio with respect to the prize—are not universally observed. Thus we attempt to provide an encompassing model that explains the conditions on contestants' preferences under which they do occur, and indeed, when this conventional wisdom is reversed. We begin with a simple Tullock share contest with a general preference structure, but then advance our framework to include a general contest success function as well as providing an analysis where the prize is endogenously determined by aggregate efforts. Throughout all advancements, we observe the rate of change of the marginal rate of substitution as pivotal to the outcome of the contest.

Our focus here is on Tullock contests in which a prize is shared, where the contest success function determines the *share* of the perfectly divisible prize a contestant receives, rather than the other 'winner-take-all' interpretation where the contest success function determines the *probability* that a player receives the prize. If each contestant is risk neutral and the cost of effort is linear the two interpretations are strategically equivalent, since then every contestants' expected payoff in a winner-take-all contest is equivalent to their payoff in a prize-sharing contest. The equivalence, however, breaks down in all but this simplest of settings and the two types of contest command separate study.

In winner-take-all contests, non-linear evaluation of the contest outcome has been considered by supposing that contestants evaluate the outcome of the contest using a (concave) utility function. This allows contestants' risk preferences to be captured, study of which has commanded substantial attention in the literature (Hillman and Katz, 1984; Long and Vousden, 1987; Skaperdas and Gan, 1995; Konrad and Schlesinger, 1997; Treich, 2010; Cornes and Hartley, 2012).[1] This formulation does not, however, carry over to the prize-sharing interpretation of contests where the evaluation of the outcome of the contest should be the share of the prize received with certainty (after accounting for the cost of effort), not a probability-weighted average of utilities in the two states that may emerge in a winner-take-all contest.

Our contribution is to model and analyze more general preferences in Tullock sharing contests that extend the domain of applicability of these important models to situations where contestants might have more than simple linear evaluation of the context outcome. This extension is not without consequence for, whilst we show that as in standard contests reasonable conditions admit a unique equilibrium, a conventional wisdom of the contest literature—that effort is increasing in the size of the prize—does not hold when preferences satisfy some very standard conditions. Understanding this is of key importance when modeling contests in which contestants might have more general preferences.

The remainder of the article is structured as follows. Section 1.1 provides an illustrative example to highlight the importance of non-linear preferences in contests. In Section 2 we outline share contests in which players have general preferences. In Section 3, we characterize

[1] Note that Long and Vousden (1987) analyzes risk aversion but within the context of a rent being shared. That is, the player can increase (or decrease) their share of a rent from an exogenously given level. For their analysis a specific form of separable preference structure must be assumed.

the Nash equilibria. Section 4 shows the influence of the size of the rent and Section 5 analyzes the dissipation ratio. Section 6 provides further extensions to general contest success functions and prizes endogenously determined by aggregate effort. Section 7 provides our concluding remarks.

1.1 An illustrative example

To provide an illustrative example, consider a simple Tullock contest for a perfectly divisible exogenously-given prize R in which all contestants are identical and have the utility function

$$u^i(z, x) = (z - k)^\alpha - cx,$$

where $z = \frac{x^i}{X}R$ is the contestant's share of the prize, x^i and X are player i's effort and aggregate effort, respectively, $\alpha \in (0, 1]$, and $k \geq 0$. Since setting $\alpha = 1$ yields the familiar contest setting with linear cost of effort, this illustrative example is well suited for exploring the implications of diminishing returns in the allocation of the prize. We can deduce that the equilibrium effort of any contestant, written as a function of R, is given by

$$x^*(R) = \frac{\alpha(n-1)R}{n^2 c}\left(\frac{R}{n} - k\right)^{\alpha-1},$$

where we assume $R/n > k$. The study of contests usually identifies a positive relationship between the size of the contested prize and equilibrium effort. Yet this may no longer hold when players' preferences are transformed from the conventional $\alpha = 1$. To see this note that:

$$x^{*\prime}(R) \gtreqless 0 \Leftrightarrow R \gtreqless \frac{kn}{\alpha}.$$

We deduce that with linear preferences ($\alpha = 1$) it is always the case that individual—and by extension (in the case of homogeneous contestants) aggregate—effort is increasing in the prize R: the conventional contest outcome holds. Interestingly, however, when $\alpha < 1$ there is a non-empty range of prizes $kn < R < kn/\alpha$ where individual—and thus aggregate—effort is *decreasing* in the prize. To illustrate the effect of allowing for diminishing marginal returns in the prize share z^i, Figure 1 plots the players' aggregate equilibrium effort against the size of the prize R for different concavity parameters: we observe that the degree of concavity determines whether aggregate effort is monotonically increasing in the size of the prize, or not.

Using this example it is clear that if the structure of players' preferences differ from the conventional structure, many of the main conclusions of contest theory may fail to hold. In this article we are interested in how contests are affected when players have alternative preference structures. As already mentioned, we show that the key to understanding contests with general preferences is the relationship between the rate of change of marginal rate of substitution between the prize share and effort.

2 Contests with general preferences

Consider a set of individual players $N = \{1, \ldots, n\}$ that participate in a contest to obtain a rent, or prize, R. Their success in the contest is determined by their effort relative to the effort of other contestants and is given by the contest success function $\phi(x^i, \mathbf{x}^{-i})$, where x^i denotes the costly effort of player $i \in N$ and $\mathbf{x}^{-i}$ denotes the vector of all other contestants' effort levels. In this article we focus on contests in which the prize is perfectly divisible and is shared between contestants in accordance with the contest success function. Define z^i as being contestant i's allocation of the prize from the contest:

$$z^i \equiv \phi(x^i, \mathbf{x}^{-i})R. \tag{1}$$

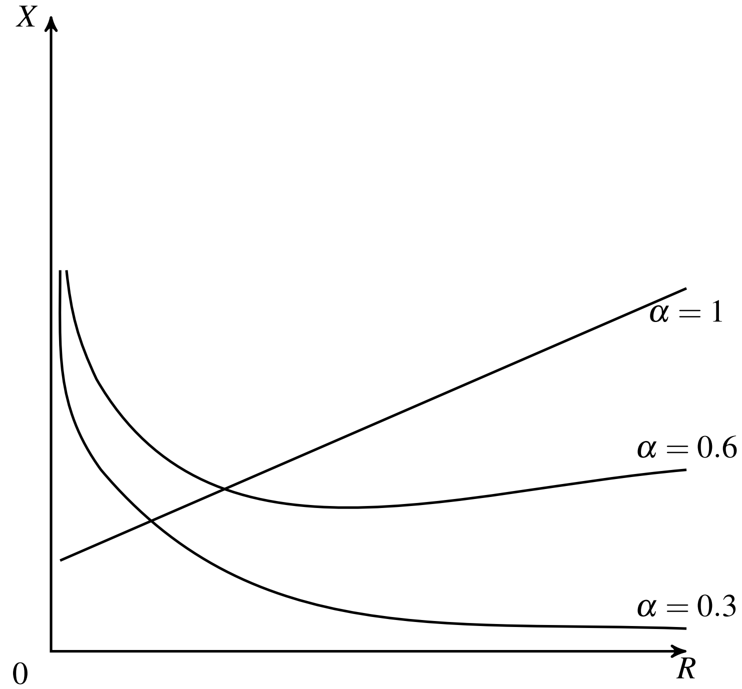

Figure 1: Aggregate equilibrium effort in contests

We begin by studying a 'simple' Tullock contest for an exogenously-given prize of size R in which

$$\phi(x^i, \mathbf{x}^{-i}) = \begin{cases} \frac{x^i}{x^i + X^{-i}} & \text{if } X > 0 \text{ or} \\ \frac{1}{n} & \text{otherwise,} \end{cases}$$

where $X \equiv \sum_{j \in N} x^j$ is the aggregate effort and $X^{-i} \equiv X - x^i$. Later in the article we consider more general contest success functions, as well as situations in which the size of the prize is influenced by the effort of contestants, i.e., where the prize is endogenously determined by contestants' efforts.

For each contestant i we define a utility function $u^i(z, x)$ over their prize allocation from the contest, z, and their effort in contesting the prize, x. We denote $MRS^i(z, x)$ as contestant i's marginal rate of substitution between z and x such that:[2]

$$MRS^i(z, x) \equiv -\frac{u^i_x}{u^i_z}.$$

Consider the (x^i, z^i)-space. Since utility is increasing in z but decreasing in effort, the indifference curves derived from the utility function defined above will have an upward slope (measured by the marginal rate of substitution just defined) and utility is increasing in a north-west direction.

We focus on cases where (heterogenous) contestants' utility is increasing in their allocation of the prize, at a decreasing rate; decreasing in effort, at an increasing rate; and if there are complementarities between the allocation of the prize from the contest and effort then these are sufficiently small.

Assumption 1. *For each $i \in N$, the utility function is differentiable as many times as required, $u^i_z > 0$, $u^i_{zz} \leq 0$, $u^i_x < 0$, $u^i_{xx} \leq 0$, and*

$$u^i_{zx} < \min\left\{ MRS^i \left| u^i_{zz} \right|, \frac{1}{MRS^i} \left| u^i_{xx} \right| \right\}.$$

[2]Throughout we use superscripts to identify contestants, and subscripts to denote partial derivatives.

Concavity of the utility function is, of course, standard. The last condition in the assumption—which ensures complementarities between z and x are sufficiently small—implies that $MRS^i_z > 0$ and $MRS^i_x > 0$; to observe this note that

$$MRS^i_z = -\frac{u^i_z u^i_{zx} - u^i_x u^i_{zz}}{(u^i_z)^2} \qquad \text{and} \qquad MRS^i_x = -\frac{u^i_z u^i_{xx} - u^i_x u^i_{zx}}{(u^i_z)^2}$$

and therefore $MRS^i_z > 0 \Leftrightarrow u^i_{zx} < -MRS^i u^i_{zz}$ and (noting that $u^i_x < 0$) $MRS^i_x > 0 \Leftrightarrow u^i_{zx} < -\frac{1}{MRS^i} u^i_{xx}$.

Assumption 1 allows for a very broad class of preference structures. For instance, this framework nests the standard linear preference structure $u^i(z,x) = z - x$, which is the dominant structure used within the contest literature. We can also capture convex costs of effort if we specified $u^i(z,x) = z - c^i(x)$ with $c^i_x > 0, c^i_{xx} > 0$. The level of generality within our framework even allows for non-separable preferences between x and z allowing us to capture situations, where, for example, the marginal value of the prize is influenced by the effort exerted in contesting the prize. Thus by considering a general preference structure, our framework can not only provide an analysis that nests previous studies of share contests but also provides a tractable methodology by which to consider alternative and novel preference structures, which can be used to advance and expand the understanding and applicability of contests.

The dominant framework for exploring sharing contests in the literature has been to assume preferences are (quasi-)linear, with utility being linear in the share of the prize received. This means that existing studies have neglected to consider both diminishing marginal utility for contestants from their allocation of the prize, and complementarities between the effort in the contest and their enjoyment of the prize. To advance the analysis of contests toward a larger class of preference structures it is imperative to consider these issues.

3 Characterizing equilibria in Tullock contests with general preferences

We now turn to characterize equilibria in a simple Tullock contest over an exogenously-given perfectly divisible prize R. We seek a Nash equilibrium in the simultaneous-move game of complete information in which the player set is the contestants $N = \{1, \ldots, n\}$; their strategies are their choice of effort $x^i \in \mathbb{R}_+$; and their payoffs are given by their utility of the contest outcome $u^i(z,x)$ where $x = x^i$ and $z = \frac{x^i}{x^i + X^{-i}} R$, which we assume satisfies Assumption 1. Note that we allow all players to be heterogeneous, and we assume $n \geq 2$.

First, we note that at a Nash equilibrium of the contest each player may be seen as solving the problem

$$\max_{x^i \in \mathbb{R}_+} u^i \left(\frac{x^i}{x^i + X^{-i}} R, x^i \right).$$

The necessary first-order condition for x^i to maximize utility given $X^{-i} = \sum_{j \neq i \in N} x^j$, i.e. identify a best response, is

$$\frac{X^{-i}}{(x^i + X^{-i})^2} R u^i_z + u^i_x \leq 0, \tag{2}$$

with equality if $x^i > 0$.

Lemma 1. *Suppose Assumption 1 is satisfied for a contestant, Then the first-order condition is both necessary and sufficient for identifying their best response.*

Proof. The second-order sufficient condition is

$$u^i_{zx} 2 \frac{X^{-i}}{(x^i + X^{-i})^2} R + u^i_{zz} \left(\frac{X^{-i}}{(x^i + X^{-i})^2} R \right)^2 + u^i_{xx} - u^i_z 2 \frac{X^{-i}}{(x^i + X^{-i})^3} R < 0.$$

For any $x^i > 0$, the first-order condition implies $\frac{X^{-i}}{(x^i+X^{-i})^2}R = -\frac{u^i_x}{u^i_z} = MRS^i$. As such, the second-order condition can be re-written

$$\begin{aligned}
&2MRS^i u^i_{zx} + MRS^i\left(MRS^i u^i_{zz} + \frac{1}{MRS^i}u^i_{xx}\right) - \frac{2}{x^i+X^{-i}}MRS^i u^i_z\\
<&2MRS^i u^i_{zx} - MRS^i\left(MRS^i|u^i_{zz}| + \frac{1}{MRS^i}|u^i_{xx}|\right)\\
<&2MRS^i u^i_{zx} - 2MRS^i \min\left\{MRS^i\left|u^i_{zz}\right|, \frac{1}{MRS^i}\left|u^i_{xx}\right|\right\}\\
=&2MRS^i\left(u^i_{zx} - \min\left\{MRS^i\left|u^i_{zz}\right|, \frac{1}{MRS^i}\left|u^i_{xx}\right|\right\}\right)\\
<&0,
\end{aligned}$$

since $u^i_{zx} < \min\left\{MRS^i\left|u^i_{zz}\right|, \frac{1}{MRS^i}\left|u^i_{xx}\right|\right\}$ under Assumption 1. $\square$

Contestant i's best response is thus given by $b^i(X^{-i};R) = \max\{0, x^i\}$ where x^i is the unique solution to

$$MRS^i\left(\frac{x^i}{x^i+X^{-i}}R, x^i\right) = \frac{X^{-i}}{(x^i+X^{-i})^2}R,$$

and we seek a Nash equilibrium in which players use mutually consistent best responses.

Rather than working directly with best responses, we turn to analyze the contest using an extension of the 'share function' approach, as first developed by Cornes and Hartley (2005), to allow for general preferences. For each contestant we define a share function that gives their share of the prize that is consistent with a Nash equilibrium in which the aggregate effort of all contestants is $X > 0$. By replacing X^{-i} with $X - x^i$ in the first-order condition (2) and letting $\sigma^i \equiv x^i/X$ we deduce that contestant i's share function is given by $s^i(X;R) = \max\{0, \sigma^i\}$ where σ^i is the solution to

$$l^i(\sigma^i, X; R) \equiv MRS^i(\sigma^i R, \sigma^i X) - (1-\sigma^i)\frac{R}{X} = 0. \tag{3}$$

Share functions shed light on individual behavior consistent with a Nash equilibrium: $Xs^i(X;R)$ is the effort of contestant i consistent with a Nash equilibrium in which the aggregate effort of all contestants is $X > 0$. To identify a Nash equilibrium, we require consistency at the aggregate level; that is, the sum of individual efforts being equal to the aggregate effort, or for the sum of individual share functions to be equal to unity. Letting

$$S(X;R) \equiv \sum_{j\in N} s^j(X;R),$$

we have the following equivalence statement.

Lemma 2. *In a contest with prize R, there is a Nash equilibrium with aggregate effort $X^* > 0$ if and only if*

$$S(X^*;R) = 1.$$

Proof. We seek to show that X^* is a Nash equilibrium if and only if $S(X^*;R) = 1$. First, the "if" part. If X^* is a Nash equilibrium then $x^{i*} = b^i(X^{-i*};R)$ for all $i \in N$. This implies $x^{i*} = b^i(X^* - x^{i*};R)$ which in turn implies $x^{i*} = X^* s^i(X^*;R)$ for all $i \in N$, and therefore that $X^* = X^*\sum_{j\in N} s^j(X^*;R)$, and consequently $S(X^*;R) = 1$. For the "only if" part, note that for each $i \in N$, $X^* s^i(X^*;R) = b^i(X^* - X^* s^i(X^*;R))$. If $S(X^*;R) = 1$ then $X^* = X^* S(X^*;R)$ and so for each $i \in N$, $X^* s^i(X^*;R) = b^i(X^* S(X^*;R) - X^* s^i(X^*;R)) = b^i(X^{-i*};R)$, thus allowing us to conclude that $x^{i*} = X^* s^i(X^*;R)$ for all $i \in N$ constitutes a Nash equilibrium. $\square$

Questions of the existence and uniqueness of Nash equilibrium now rest on consideration of the behavior of the aggregate share function $S(X;R)$, whose properties are derived from individual share functions, and its intersection with the unit line. The following proposition sets out the properties of individual share functions.

Proposition 1. *For each $i \in N$,*

1. *$s^i(X;R)$ is a continuous function defined for all $X > 0$ and R.*
2. *$s^i(X;R) \to 1$ as $X \to 0$ and either $s^i(X;R) = 0$ for all $X \geq \bar{X}^i(R) \equiv R/MRS^i(0,0)$ if $MRS^i(0,0) > 0$ or, if $MRS^i(0,0) = 0$, $s^i(X;R) \to 0$ as $X \to \infty$.*
3. *$s^i(X,R)$ is strictly decreasing in X for $0 < X < \bar{X}^i(R)$.*

Proof. 1. Recall from (3) that a contestant's share function is implicitly defined as the value of σ^i where

$$l^i(\sigma^i, X; R) \equiv MRS^i(\sigma^i R, \sigma^i X) - (1-\sigma^i)\frac{R}{X} = 0,$$

if σ^i is positive, otherwise the share function takes the value zero. Continuity of the share function is established by the assumed smooth nature of the functions in the first-order condition. Next, note that

$$l^i_\sigma = R\, MRS^i_z + X\, MRS^i_x + \frac{R}{X} > 0, \tag{4}$$

allowing us to conclude that there is at most one value of $\sigma^i > 0$ where $l^i(\sigma^i, X; R) = 0$, so $s^i(X;R)$ is a function.

2. When $\sigma^i = 0$, $l^i(0, X; R) = MRS^i(0,0) - R/X$. The fact just deduced that $l^i_\sigma > 0$ implies that if $l^i(0,X;R) \geq 0$ then $l^i(\sigma^i, X; R) > 0$ for all $\sigma^i > 0$, and therefore $s^i(X;R) = 0$. If $MRS^i(0,0) > 0$, $\bar{X}^i(R) \equiv R/MRS^i(0,0)$ is well-defined and we can conclude that $s^i(X,R) = 0$ for all $X \geq \bar{X}^i(R)$. If $MRS^i(0,0) = 0$ then $l^i(0,X;R) = -R/X$. As $X \to \infty$, $l^i(0,X;R) \to 0$ and then the fact that $l^i_\sigma > 0$ implies $s^i(X;R) \to 0$.

As $X \to 0$, $Xl^i(\sigma^i, X; R) = XMRS^i(\sigma^i R, \sigma^i X) - (1-\sigma^i)R \to -(1-\sigma^i)R$, so $\sigma^i = 1$ is the only possibility to achieve $l^i(\sigma^i, X; R) = 0$, implying $s^i(X;R) \to 1$.

3. Finally, to understand how share functions vary with X we apply implicit differentiation to (3) to deduce that

$$s^i_X = -\frac{l^i_X}{l^i_\sigma} = -\frac{\sigma^i MRS^i_x + (1-\sigma^i)\frac{R}{X^2}}{R\, MRS^i_z + X\, MRS^i_x + \frac{R}{X}} < 0, \tag{5}$$

confirming the strict monotonicity. □

The properties of individual share functions imply that in a contest with prize R the aggregate share function $S(X;R)$, being constructed from a sum of at least two individual share functions, exceeds 1 when X is small enough, is less than one when X is large enough, and is continuous and strictly decreasing in X implying there is exactly one value of X where $S(X;R) = 1$.

Proposition 2. *In a contest with prize R there is a unique Nash equilibrium with aggregate effort X^* such that*

$$S(X^*;R) = 1$$

in which the equilibrium effort of contestant i is $x^i = X^* s^i(X^*;R)$.

Proof. From Lemma 2 we know that Nash equilibria are identified by intersections of $S(X;R)$ with the unit line. From Proposition 1 we also know that individual share functions are single-valued, continuous and strictly decreasing in $X > 0$, and have the property $s^i(X;R) \to 1$ as $X \to 0$ and either $s^i(X;R) = 0$ for all $X \geq R/MRS^i(0,0)$ or, if $MRS^i(0,0) = 0$, that $s^i(X;R) \to 0$ as $X \to \infty$. As such, $S(X;R) \to n$ as $X \to 0$ and (at worst) $S(X;R) \to 0$ when $X \to \infty$.

Combined with the fact that $S(X;R)$ is continuous and strictly decreasing in $X > 0$, this implies there is a single value of X where $S(X;R) = 1$, and so the Nash equilibrium is unique. $\square$

As such, we confirm that in prize-sharing contests where players can have more general preferences over their allocation of the prize and the effort exerted in contesting the prize, the uniqueness of Nash equilibrium—as found in simple Tullock contests—is preserved under our stated assumptions.

4 The effect of the size of the contested prize

We now turn to investigate how contestants' equilibrium behavior depends on the size of the prize they are contesting. We write $\mathcal{X}(R)$ for the equilibrium aggregate effort in a contest where the size of the prize is R, which is implicitly defined by

$$S(\mathcal{X}(R);R) = 1. \tag{6}$$

To determine the sign of $\mathcal{X}'(R)$, we make use of expression (6) to deduce:[3]

$$\mathcal{X}'(R) = -\frac{\sum_{j\in N} s^j_R}{\sum_{j\in N} s^j_X} \tag{7}$$

Having already found that $s^i_X < 0$ for all $i \in N$ (Proposition 1), how equilibrium aggregate effort responds to a change in the size of the prize will rely on the features of s^j_R, that we turn to investigate next.

Lemma 3.

$$s^i_R \gtreqless 0 \Leftrightarrow z^i MRS^i_z - MRS^i \lesseqgtr 0.$$

Proof. Recall from (3) that a contestant's share function is implicitly defined as the value of σ^i where

$$l^i(\sigma^i, X; R) \equiv MRS^i(\sigma^i R, \sigma^i X) - (1-\sigma^i)\frac{R}{X} = 0.$$

As such,

$$\begin{aligned} s^i_R &= -\frac{l^i_R}{l^i_\sigma} \\ &= -\frac{\sigma^i MRS^i_z - (1-\sigma^i)\frac{1}{X}}{R\, MRS^i_z + X\, MRS^i_x + \frac{R}{X}}. \end{aligned}$$

The denominator (as deduced in (4)) is positive. Noting that $\sigma^i R = z^i$ and that $(1-\sigma^i)\frac{R}{X} = MRS^i$ from the first-order condition, gives

$$s^i_R = -w^i(z^i MRS^i_z - MRS^i),$$

where $w^i = (R(R\, MRS^i_z + X\, MRS^i_x + \frac{R}{X}))^{-1} > 0$, from whence the statement in the lemma follows. $\square$

[3]Strictly speaking, we should not implicitly differentiate (6) since whilst it is continuous, where $X = \bar{X}^i(R)$ for a contestant that contestant drops out of the sum so there will be a kink in the function. Nevertheless, the approach will be used since it is simple and has intuitive merit, and is of course valid so long as we consider only neighborhoods in which there is no such $\bar{X}^i(R)$. Where this is not the case, the proof of exactly the same result can be made by contradiction.

Putting the expression for $\mathcal{X}'(R)$ together with the expression for s^i_R, it follows that

$$\text{sgn}\{\mathcal{X}'(R)\} = -\,\text{sgn}\{\sum_{j\in N} w^j(z^j MRS^j_z - MRS^j)\}.$$

This allows us to draw the following conclusion regarding how the equilibrium aggregate effort in a contest changes with the size of the prize, which differs significantly from the standard conclusion for prize-sharing contests (that equilibrium aggregate effort always increases in R).

Proposition 3. *Suppose the preferences of all contestants satisfy Assumption 1. If $\sum_{j\in N} w^j(z^j MRS^j_z - MRS^j) \gtreqless 0$ then $\mathcal{X}'(R) \lesseqgtr 0$. A sufficient condition for this is for $\frac{MRS^i}{z^i}$ to be increasing (constant, decreasing) in z^i for all contestants.*

This proposition reveals that how aggregate effort changes with the size of the prize is crucially dependent on how the ratio MRS^i/z^i changes with the allocation of the prize (since the sign of the derivative of this object is equal to the sign of $z^i MRS^i_z - MRS^i$). If MRS^i/z^i is decreasing in z^i for all contestants—as assumed in the contest literature so far as the marginal rate of substitution is constant in z^i—then the equilibrium aggregate effort of contestants is always increasing in the size of the contested prize. However, with our more general preferences MRS^i is increasing (weakly) in z, and if it increases sufficiently so that MRS^i/z^i also increases then aggregate effort may, in fact, *decrease* with a larger prize. This was the case in the motivating example at the start of the article and means that if we account for contestants having more general preferences in prize-sharing contests we must expel the conventional wisdom that larger prizes always command greater effort.

4.1 Interpreting the result

In a contest player i can be seen as solving the following constrained optimization problem:

$$\max_{x^i\in\mathbb{R}_+} u^i(z^i, x^i) \text{ s.t. } z^i = \frac{x^i}{\sum_{j\in N} x^j} R$$

Put differently, player i maximizes a utility function that is increasing in her share of the contested prize, z^i, and decreasing in contest effort x^i, subject to the constraint that implies her share of the contested prize is increasing in contest effort. This constraint can be interpreted as a budget constraint since it maps the combinations (z^i, x^i) that are achievable for a given R.

We have shown above that optimizing yields:

$$MRS^i = \frac{X - x^i}{X^2} R,$$

where the (positive) marginal rate of substitution equals the amount by which the budget constraint is being relaxed when contest effort increases. Bearing in mind that the marginal rate of substitution captures the relative increase of utility of marginally increasing the share of the contested prize enjoyed to marginally reducing the contest effort, we thus require that this relative increase in utility be equal to the relative relaxation of the budget constraint. This line of reasoning allows us to visualize the condition in (x^i, z^i)-space where we represent player i's (upward-sloping) indifference curves and budget constraint. Accordingly, at optimality, for player i to be attaining the highest indifference curve for a given budget constraint, the slopes of these two loci are equal, as illustrated in Figure 2, where point a depicts the tangency between the budget constraint and an indifference curve.

Increasing the value of the prize (to R') relaxes the budget constraint since it enables any player i to increase z^i, while keeping x^i constant. Since the slope of the BC equals $\frac{X-x^i}{X^2}R$, an increase is R will translate to an increase of this slope, which is equivalent to a relaxation of

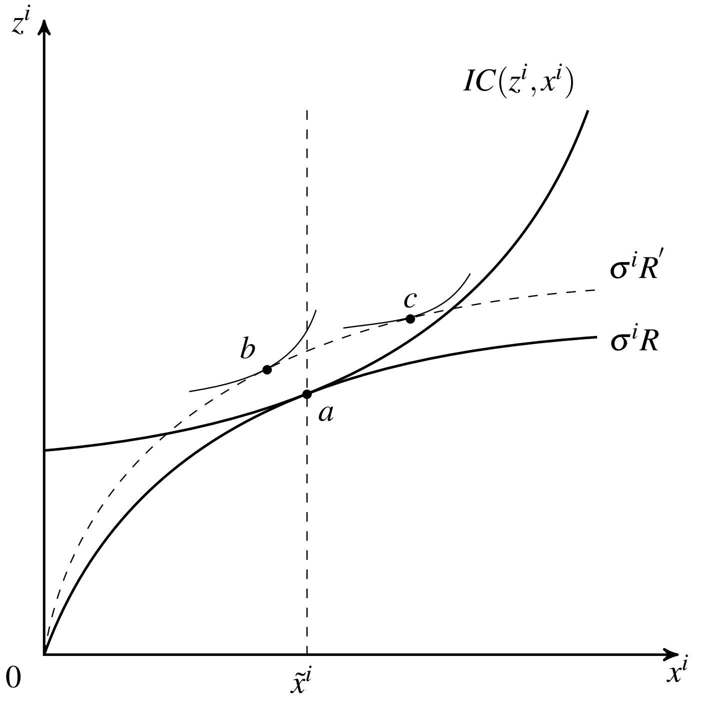

Figure 2: Equilibrium effort

the BC. To determine whether player i's effort will increase or decrease with a change in R, it is thus sufficient to look at the change of the slope of the indifference curve, when increasing R while maintaining x^i constant. Should this slope increase by more than the slope of budget constraint, this will imply that with a higher value of the prize R, player i will be able to increase his consumption of z^i while expending *less* contest effort. Should the opposite hold, at the new equilibrium more contest effort will be exerted by player i. Mathematically, the relative change in slopes is such that we witness a decrease of contest effort at equilibrium if:

$$\frac{\partial MRS^i(\sigma^i R, x^i)}{\partial R} > \frac{\partial \frac{X-x^i}{X^2} R}{\partial R} \Leftrightarrow \sigma^i MRS^i_z > \frac{X - x^i}{X^2}.$$

Utilizing (2), we deduce that $\frac{X-x^i}{X^2} = MRS^i/R$ and therefore we can re-write the above expression (noting that $\sigma^i R = z^i$) as

$$MRS^i - z^i MRS^i_z < 0, \tag{8}$$

which is exactly the same condition driving the result of Proposition 3.

Referring back to Figure 2, if an increase in the value of the prize R is accompanied by inequality (8) being satisfied, this will lead to a reduction of individual contest effort as reflected by point b. If on the other hand, the inequality is violated, then individual effort will increase, as reflected by point c.

Thus, we find that an increase in the value of the prize in a Tullock share-contest with general preferences may result in increases or decreases of individual contestants' efforts, depending on their preferences. To better understand why the literature unambiguously identifies a positive relationship, we further explore inequality (8) by computing the components of the object on the left:

$$MRS^i - z^i MRS^i_z = \frac{u^i_x u^i_z - z^i \left(u^i_x u^i_{zz} - u^i_z u^i_{zx}\right)}{\left(u^i_z\right)^2} \tag{9}$$

The bulk of the literature considers contestants with (quasi-)linear preferences similar to our motivating example when $\alpha = 1$. In other words, contestants derive a linear utility from their share if the prize, while the cost of effort is assumed to be (weakly) convex: $u^i(z, x) = z - c^i(x)$. Imposing such assumptions implies that $u^i_{zz} = u^i_{zx} = 0$ and $u^i_{xx} \leq 0$, so that the sign of expression (9) is unambiguously positive. Accordingly this implies a positive relationship between the value of the prize and individual contest effort.

From Condition (9), we realize that additive separability ($u^i_{zx} = 0$) is not the defining factor behind the monotonic relationship between prize-value and contest effort found in the literature. Indeed, diminishing marginal returns in z^i ($u^i_{zz} < 0$) proves sufficient to yield a non-standard relationship between the value of the prize and effort even with additively separable utility functions.

On the other hand, neither are diminishing marginal returns on z^i a necessary condition for obtaining a non-monotonic relationship between prize-value and contest effort. Consider Condition (9) once more, and set $u^i_{zz} = 0$. In the presence of sufficiently strong substitutability between z^i and x^i, increases in R may generate decreases in contest effort. Strong substitutability implies that higher contest effort strongly reduces the marginal utility of the prize-share.

5 The dissipation ratio

We now consider how the dissipation ratio alters under a Tullock contest with general preferences. Recall that the share function satisfies the first-order condition

$$MRS^i(\sigma^i R, \sigma^i X) = (1 - \sigma^i)\frac{R}{X},$$

and let $D = \frac{X}{R}$ be the dissipation ratio. Noting that $X = D \cdot R$, we can write the share function as $s^i(DR, R)$ which will satisfy

$$l^i(\sigma^i, DR; R) \equiv MRS^i(\sigma^i R, \sigma^i DR) - (1 - \sigma^i)\frac{1}{D} = 0.$$

Then the equilibrium dissipation ratio, written $\mathcal{D}(R)$, must satisfy

$$\sum_{j \in N} s^j(\mathcal{D}(R)R, R) = 1.$$

How the dissipation ratio changes in the size of the prize can be derived as follows:

$$\mathcal{D}'(R) = -\frac{\sum_{j \in N} \frac{\mathrm{d}s^i}{\mathrm{d}R}}{\sum_{j \in N} \frac{\mathrm{d}s^i}{\mathrm{d}D}}$$

(notice that the change from partial derivatives as, in particular, R enters both arguments of the share function). Now,

$$\frac{\mathrm{d}s^i}{\mathrm{d}R} = -\frac{\frac{\mathrm{d}l^i}{\mathrm{d}R}}{l^i_\sigma} \text{ and } \frac{\mathrm{d}s^i}{\mathrm{d}D} = -\frac{\frac{\mathrm{d}l^i}{\mathrm{d}D}}{l^i_\sigma}.$$

We deduce that

$$\begin{aligned} l^i_\sigma &= \frac{1}{\sigma^i}\left(z^i MRS^i_z + x^i MRS^i_x + \frac{\sigma^i}{D}\right), \\ \frac{\mathrm{d}l^i}{\mathrm{d}D} &= \sigma^i R \cdot MRS^i_x + \frac{1 - \sigma^i}{D^2} \text{ and} \\ \frac{\mathrm{d}l^i}{\mathrm{d}R} &= \frac{1}{R}(z^i MRS^i_z + x^i MRS^i_x). \end{aligned}$$

Under Assumption 1, each of these expressions is non-negative, and therefore we deduce that $\mathcal{D}'(R) \leq 0$, and is exactly zero only when $z^i MRS^i_z + x^i MRS^i_x = 0$ for all contestants, as in the case of linear preferences. This confirms that, whilst aggregate effort might be increasing in the size of the prize in our general case, it does not increase at a rate such that $\mathcal{D}(R) = \mathcal{X}(R)/R$ also increases. In Tullock contests with linear preferences the dissipation ratio is constant; our result confirms that under more general preferences the same result as in contests with convex costs—that the dissipation ratio decreases in the size of the prize—also holds. Our findings thus complement the literature that attempts to explain the existence of under dissipation, such as justified by loss aversion (Cornes and Hartley, 2003) and behavioral considerations (Baharad and Nitzan, 2008).

6 Extensions

In this section we pursue two generalizations of our model of contests with general preferences: the first allows for a more general contest success function; and the second allows for the prize to be endogenously determined by contestants' efforts.

6.1 General contest success function

So far, we have considered a simple Tullock contest in which the contest success function has taken the form $\phi(x^i, \mathbf{x}^{-i}) = \frac{x^i}{x^i + X^{-i}}$. Of course, Tullock contests can be more general than this, typically considering an additional parameter r, and specifying $\phi(x^i, \mathbf{x}^{-i}) = \frac{(x^i)^r}{(x^i)^r + \sum_{j \neq i \in N}(x^j)^r}$. To capture this, we will follow Cornes and Hartley (2005) and specify that

$$\phi(x^i, \mathbf{x}^{-i}) = \frac{P^i(x^i)}{\sum_{i \in N} P^i(x^i)}, \tag{10}$$

where we need to assume that $P^{i'} > 0$ and $P^{i''} \leq 0$.[4]

We now need to re-consider our analysis with this more general contest success function. According to (1), $z^i = \frac{P^i(x^i)}{\sum_i P^i(x^i)} R$. So that the share function approach can be utilized, let us change the variable of consideration and rather than focus on effort, x^i, let us think of contestants choosing what Cornes and Hartley (2005) call the contestant's "input" $y^i = P^i(x^i)$, from which effort can be derived since $x^i = P^{i^{-1}}(y^i)$ and the contest 'technology' is assumed to be monotonic so the inverse of $P^i(\cdot)$ is a singleton. With this change of variables, contestants can be seen as choosing their input to maximize their payoff $u^i(z^i, P^{i^{-1}}(y^i))$, where their share of the prize is $z^i = \frac{y^i}{Y} R$ (Y being the aggregate input $\sum_{j \in N} y^j$).

The first-order condition of this optimization problem that characterizes a contestant's input best response is

$$\frac{Y^{-i}}{(y^i + Y^{-i})^2} R u_z^i + P^{i-1} u_x^i \leq 0$$

with equality if $y^i > 0$. Replacing Y^{-i} with $Y - y^i$ and letting $\hat{\sigma}^i = y^i/Y$, this can be used to define the contestant's share function as $\hat{s}^i(Y; R) = \max\{0, \hat{\sigma}^i\}$ where, making the arguments of functions explicit, $\hat{\sigma}^i$ is the solution to

$$\hat{l}^i(\hat{\sigma}^i, Y; R) \equiv MRS^i(\hat{\sigma}^i R, P^{i-1}(\hat{\sigma}^i Y)) P^{i-1}(\hat{\sigma}^i Y) - (1 - \hat{\sigma}^i)\frac{R}{Y} = 0. \tag{11}$$

As with our previous analysis, we can use share functions to shed light on the properties of Nash equilibrium in the contest, since there is a Nash equilibrium with aggregate input Y^* if and only if $\sum_{j \in N} \hat{s}^j(Y^*; R) = 1$, in which contestant i's input is $y^{i*} = Y^* \hat{s}^i(Y^*; R)$ and therefore their contest effort is $x^{i*} = P^{i-1}(Y^* \hat{s}^i(Y^*; R))$.

[4]This implies that our CSF tracks the general function axiomatized by Skaperdas (1996) up to the difference that we impose decreasing marginal returns.

As with our simple contest, we can deduce that individual share functions $\hat{s}^i(Y;R)$ are indeed functions that are continuous and strictly decreasing on $Y > 0$. To deduce this, note that

$$\hat{l}^i_{\hat{\sigma}} = (R \cdot MRS^i_z + YP^{i-1\prime}MRS^i_x)P^{i-1} + MRS^i YP^{i-1\prime} + \frac{R}{Y} > 0$$

and

$$\hat{l}^i_Y = P^{i-1}\hat{\sigma}^i P^{i-1\prime}MRS^i_x + MRS^i\hat{\sigma}^i P^{i-1\prime} + (1-\hat{\sigma}^i)\frac{R}{Y^2} > 0,$$

so the analog of Proposition 1 follows for this case.

Then let us define the aggregate equilibrium input as a function of the contested prize as $\mathcal{Y}(R)$ which will satisfy

$$\sum_{j\in N} \hat{s}^j(\mathcal{Y}(R);R) = 1.$$

Then we turn to consider how the aggregate input varies with the size of the contested prize, noting (with our usual apology for using implicit differentiation at parts of the domain where we should not) that

$$\mathcal{Y}'(R) = -\frac{\sum_{j\in N}\hat{s}^j_R}{\sum_{j\in N} s^j_Y},$$

and therefore

$$\operatorname{sgn}\{\mathcal{Y}'(R)\} = \operatorname{sgn}\left\{\sum_{j\in N}\hat{s}^j_R\right\}.$$

For each contestant i, $\hat{s}^i_R = -\hat{l}^i_R/\hat{l}^i_{\hat{\sigma}}$ and therefore $\operatorname{sgn}\{\hat{s}^i_R\} = -\operatorname{sgn}\{\hat{l}^i_R\}$. Now,

$$\begin{aligned}\hat{l}^i_R &= \hat{\sigma}^i MRS^i_z P^{i-1} - (1-\hat{\sigma}^i)\frac{1}{Y}\\ &= \frac{1}{R}\left(\hat{\sigma}^i R \cdot MRS^i_z P^{i-1} - (1-\hat{\sigma}^i)\frac{R}{Y}\right)\\ &= \frac{P^{i-1}}{R}(z^i MRS^i_z - MRS^i),\end{aligned}$$

where the last line exploits the first-order condition. As such,

$$\operatorname{sgn}\{\hat{s}^i_R\} = \operatorname{sgn}\{MRS^i - z^i MRS^i_z\},$$

and therefore we can conclude that

$$\mathcal{Y}'(R) \lesseqqgtr 0 \Leftarrow MRS^i - z^i MRS^i_z \lesseqqgtr 0\,, \forall i \in N$$

which is implied by the ratio MRS^i/z^i being increasing (constant, decreasing) in z^i.

As such, we conclude that the aggregate *input* into the contest can vary with the size of the prize in much the same way as the aggregate effort in a simple Tullock contest. However, we wish to make conclusions about the aggregate effort in this more general case, which cannot be directly deduced from the aggregate input. Since $X = \sum_{j\in N} x^j$ where $x^i = P^{i-1}(y^i)$, to be sure of the change in aggregate effort, we need to make sure each individual contestant's effort changes in the same direction, which requires each contestant's input to change in the same direction. Of course, if we assume homogeneous contestants this is obvious, but we do not do so.

As we will show, however, if we assume the condition on the ratio of the marginal rate of substitution to prize share holds for *all* contestants we can make conclusions about the change in aggregate effort. So, suppose that $MRS^i - z^i MRS^i_z < 0$ for all $i \in N$, then we have $\mathcal{Y}'(R) < 0$, our counter-intuitive case.

Now consider how *individual* inputs change. Writing $\hat{y}^i(\mathcal{Y}(R);R) = \mathcal{Y}(R)\hat{s}^i(\mathcal{Y}(R);R)$, we find that

$$\begin{aligned}\frac{\mathrm{d}\hat{y}^i(\mathcal{Y}(R);R)}{\mathrm{d}R} &= \hat{s}^i\mathcal{Y}' + \mathcal{Y}(\hat{s}^i_Y\mathcal{Y}' + \hat{s}^i_R) \\ &= \mathcal{Y}'(\hat{s}^i + \mathcal{Y}\hat{s}^i_Y) + \mathcal{Y}\hat{s}^i_R.\end{aligned}$$

Since $MRS^i - z^iMRS^i_z < 0$ and $\mathcal{Y}' < 0$, we will have $\frac{\mathrm{d}\hat{y}^i(\mathcal{Y}(R);R)}{\mathrm{d}R} \lesseqgtr 0$ if $\hat{s}^i + \mathcal{Y}\hat{s}^i_Y > 0$. Since $\hat{s}^i_Y < 0$ this is not obvious, but if we decompose this expression, we can indeed conclude that it is true. For,

$$\begin{aligned}\hat{\sigma}^i + Y\hat{s}^i_Y &= \hat{\sigma}^i - \frac{Y\hat{\sigma}^iP^{i-1\prime}(p^{i-1}MRS^i_x + MRS^i)_(1-\hat{\sigma}^i)\frac{R}{Y}}{P^{i-1}(R\cdot MRS^i_z + YP^{i-1\prime}MRS^i_x) + MRS^iYP^{i-1\prime} + \frac{R}{Y}} \\ &= \frac{\hat{\sigma}^iRMRS^i_zP^{i-1} + \hat{\sigma}^i\frac{R}{Y} - (1-\hat{\sigma}^i)\frac{R}{Y}}{P^{i-1}(R\cdot MRS^i_z + YP^{i-1\prime}MRS^i_x) + MRS^iYP^{i-1\prime} + \frac{R}{Y}}\end{aligned}$$

by putting terms over a common denominator and cancelling. Now, the first-order condition can be exploited to reduce the numerator of this expression to

$$P^{i-1\prime}(z^iMRS^i_z - MRS^i) + \hat{\sigma}^i\frac{R}{Y}$$

which is positive when $MRS^i - z^iMRS^i_z < 0$. Combined with a positive denominator, this allows us to conclude that when $MRS^i - z^iMRS^i_z < 0$ for all $i \in N$, not only is $\mathcal{Y}'(R) < 0$ but $\frac{\mathrm{d}\hat{y}^i(\mathcal{Y}(R);R)}{\mathrm{d}R} < 0$ for all $i \in N$ which implies that each individual contestant's effort will be lower, and therefore aggregate effort will reduce.

6.2 Endogenous rent

We now turn to consider the case where the contested rent is not fixed, but is influenced by the effort of contestants. So that we can consider different rent-generating technologies and understand the effect on equilibrium effort from contestants, we define the rent as $R = f(X,\alpha)$ where α is interpreted as a productivity parameter and is such that $f_\alpha > 0$. Whilst we allow for the rent to be increasing or decreasing in aggregate effort[5], we do impose the following assumption.

Assumption 2. *$f(X,\alpha)$ is a continuous function that is differentiable as many times as required and $f_{XX} \leq 0$. We also impose the technical assumptions $\lim_{X\to\infty} f_X \not> 0$; $|f_{XX}|$ is finite $\forall X$; and, if $f_X < 0$ then for each contestant for whom $MRS^i_z > 0$, $f_X > \frac{f_{XX} - MRS^i_x}{MRS^i_z}$.*

According to (1), $z^i = \frac{x^i}{x^i+X^{-i}}f(x^i + X^{-i};\alpha)$ and the first-order condition that will be used to identify a contestant's best response is:

$$u^i_z\left[\frac{x^{-i}}{(x^i+X^{-i})^2}f(x^i+X^{-i};\alpha) + \frac{x^i}{x^i+X^{-i}}f_X(x^i+X^{-i};\alpha)\right] + u^i_x \leq 0 \tag{12}$$

with equality if $x^i > 0$.

In the next lemma we demonstrate the concavity of the optimization problem contestants may be seen as facing.

Lemma 4. *Suppose Assumptions 1 and 2 are satisfied. Then the first-order condition is both necessary and sufficient for identifying a contestant's best response.*

[5]Not restricting f_X to be positive enables us to capture with our model the case of Cournot competition where $f = XP(X)$, which is increasing where demand is inelastic and decreasing where it is elastic.

Proof. Omitting the arguments of functions and using the simplifying notation $X = X^{-i} + x^i$ and $\sigma^i = x^i/X$, the second derivative of the objective function is

$$u^i_{zz}\left[\frac{(1-\sigma^i)f}{X} + \sigma^i f_X\right]^2 + u^i_{xx} + 2u^i_{zx}\left[(1-\sigma^i)\frac{f}{X} + \sigma^i f_X\right] + u^i_z\left[-\frac{2(1-\sigma^i)}{X}\left(\frac{f}{X} - f_X\right) + \sigma^i f_{XX}\right]$$

Since function $f(X;\alpha)$ is concave, $f/X > f_X$, $\forall X > 0$, thus implying that the forth term of the above expression is necessarily negative. It is thus sufficient to show that:

$$u^i_{zz}\left[(1-\sigma^i)\frac{f}{X} + \sigma^i f_X\right]^2 + u^i_{xx} + 2u^i_{zx}\left[(1-\sigma^i)\frac{f}{X} + \sigma^i f_X\right] < 0$$

Since the first-order condition implies $MRS^i = (1-\sigma^i)\frac{f}{X} + \sigma^i f_X$, the above expression can be written as:

$$u^i_{zz}\left[MRS^i\right]^2 + u^i_{xx} + 2u^i_{zx}MRS^i < 0$$

Following the steps of the proof of Lemma 1, we deduce that this inequality is satisfied given Assumption 1. ☐

Adopting the same approach as in the rest of the paper, contestant i's share function in this contest, that we write $\tilde{s}^i(X;\alpha)$, is given by $\tilde{s}^i(X;\alpha) = \max\{0, \sigma^i\}$, where σ^i is the solution to:

$$\tilde{l}^i(\sigma^i, X;\alpha) \equiv MRS^i(\sigma^i f(X;\alpha), \sigma^i X) - (1-\sigma^i)\frac{f(X;\alpha)}{X} - \sigma^i f_X(X;\alpha) = 0. \tag{13}$$

We seek a Nash equilibrium of the contest that we identify by the aggregate effort X^* such that $\sum_{j\in N}\tilde{s}^j(X^*;\alpha) = 1$.

Following the reasoning of our previous analysis, we can establish the analog of Proposition 1 that elucidated the properties of share functions for the standard case.

Proposition 4. *Suppose Assumptions 1 and 2 are satisfied. Then for each contestant $i \in N$,*

1. *$\tilde{s}^i(X;\alpha)$ is a function defined for all $X > 0$ and α;*
2. *$\tilde{s}^i(X;\alpha) \to 1$ as $X \to 0$, and either $\tilde{s}^i(X;\alpha) = 0$ for all $X \geq \bar{X}^i(\alpha)$ where $\bar{X}^i(\alpha)$ is such that $MRS^i(0,0) = f(\bar{X}^i(\alpha);\alpha)/\bar{X}^i(\alpha)$ if $MRS^i(0,0) > 0$, or if $MRS^i(0,0) = 0$, $\tilde{s}^i(X;\alpha) \to 0$ as $X \to \infty$.*
3. *$\tilde{s}^i(X;\alpha)$ is strictly decreasing in X for $0 < X < \bar{X}^i(\alpha)$.*

Proof. 1. Note that:

$$\tilde{l}^i_\sigma = f(X;\alpha)MRS^i_z + XMRS^i_x + \frac{f(X;\alpha)}{X} - f_X(X;\alpha) > 0$$

where the sign follows from the concavity of $f(X;\alpha)$ which implies $f(X;\alpha)/X > f_X(X;\alpha)$. This in turn implies that there is at most one value of σ^i where $\tilde{l}^i(\sigma^i, X;\alpha) = 0$, so $\tilde{s}^i(X;\alpha)$ is a function.

2. When $\sigma^i = 0$, $\tilde{l}^i = MRS^i(0,0) - \frac{f(X;\alpha)}{X}$. Since $\tilde{l}^i_\sigma > 0$, if $\tilde{l}^i(0, X;\alpha) \geq 0$, then $\tilde{l}^i(\sigma^i, X;\alpha) > 0$ for all $\sigma^i > 0$, and therefore $\tilde{s}^i(X;\alpha) = 0$. As such, if $MRS^i(0,0) > 0$, defining $\bar{X}^i(\alpha)$ as in the proposition allows us to define $\tilde{s}^i(X;\alpha) = 0$ for all $X \geq \bar{X}^i(\alpha)$. If $MRS(0,0) = 0$, then $\tilde{l}^i(0, X;\alpha) = -\frac{f(X;\alpha)}{X}$. As $X \to \infty$, and since $\lim_{X\to\infty} f_X \not> 0$, $f(x;\alpha)/X \to 0$ and the fact that $\tilde{l}^i_\sigma > 0$ implies $\tilde{s}^i(X;\alpha) \to 0$.

As $X \to 0$, we show that $X\tilde{l}^i(\sigma^i, X;\alpha) \to 0$ when $\sigma^i = 1$. Consider the following expression:

$$X\tilde{l}^i(\sigma^i, X;\alpha) = X\left[MRS^i(\sigma^i f(X;\alpha), \sigma^i X) - \sigma^i\frac{f(X;\alpha)}{X}\right] + X\left[\frac{f(X;\alpha)}{X} - \sigma^i f_X(X;\alpha)\right]$$

This is composed of two terms. Focusing on the first term, as $X \to 0$, since $f(X;\alpha) \to 0$, the entire term tends to zero. Looking next at the second term. One candidate for satisfying $X\tilde{l}^i(\sigma^i, X;\alpha) = 0$ is that $\lim_{X\to 0}\{f/X - \sigma^i f_X\} = 0$. Showing this is equivalent to showing the following:

$$\lim_{X\to 0} \frac{\frac{f(X;\alpha)}{X}}{\sigma^i f_X} = \lim_{X\to 0} \frac{f(X;\alpha)}{\sigma^i X f_X} = 1 \tag{14}$$

Since the limit of this expression is indeterminate, we apply l'Hospital's rule to obtain:

$$\lim_{X\to 0} \frac{f(X;\alpha)}{\sigma^i X f_X} = \lim_{X\to 0} \frac{f_X}{\sigma^i \left[f_X + X f_{XX}\right]}$$

Since $f_{XX} \leq 0$ and is finite, we conclude that:

$$\lim_{X\to 0} \frac{f(X;\alpha)}{\sigma^i X f_X} = \frac{1}{\sigma^i}$$

Thus implying that for equation (14) to be satisfied we require $\sigma = 1$. Since $\tilde{l}^i_\sigma > 0$, there exists at most one value of σ^i satisfying the equality, thus implying that $\tilde{s}^i \to 1$.

3. Applying implicit differentiation to (13) we deduce that:

$$\tilde{s}^i_x = -\frac{\tilde{l}^i_x}{\tilde{l}^i_\sigma} = -\frac{\sigma^i f_X MRS^i_z + \sigma MRS^i_x - (1-\sigma^i)\left(\frac{Xf_X - f}{X^2}\right) - \sigma f_{XX}}{f MRS^i_z + X MRS^i_x + \frac{f}{X} - f_X} \tag{15}$$

Since the denominator is positive, to establish the sign of (15) we thus need to determine the sign of the numerator, showing it is positive. Note first that $-(1-\sigma^i)\left(\frac{Xf_X - f}{X^2}\right) = \frac{1-\sigma^i}{X}\left(\frac{f}{X} - f_X\right) > 0$ by concavity of f. Now, since f_X is allowed to take negative values, for (15) to be positive it is sufficient to establish the numerator is positive for the case where $\sigma^i = 1$, i.e. the case where the highest weight is being allocated to f_X. We thus need to ensure that:

$$f_X MRS^i_z + MRS^i_x - f_{XX} > 0 \Rightarrow f_X > \frac{f_{XX} - MRS^i_x}{MRS^i_z},$$

which is assumed of the rent-generation technology. □

We now explore the effect of productivity on individual and aggregate contest effort. In a contest with endogenous prize determination, the aggregate effort consistent with a Nash equilibrium is given by $\tilde{\mathcal{X}}(\alpha)$ which is defined such that

$$\sum_{j\in N} \tilde{s}^j(X;\alpha) = 1.$$

Applying implicit differentiation (again with apology) we deduce that the effect of a change in the way rent is generated on equilibrium aggregate effort is given by

$$\tilde{\mathcal{X}}'(\alpha) = -\frac{\sum_{j\in N} \tilde{s}^j_\alpha}{\sum_{j\in N} \tilde{s}^j_X}.$$

Having already shown that the denominator is negative in the previous proposition, we deduce that the sign of this expression is equal to the sign of the numerator. The next lemma establishes the condition determining how individual shares vary with the productivity parameter:

Lemma 5. *Suppose Assumptions 1 and 2 are satisfied. Then*

$$\tilde{s}^i_\alpha \gtreqless 0 \Leftrightarrow \frac{1}{f}\left(f_\alpha(z^i MRS^i_z - MRS^i) + \sigma^i(f_\alpha f_X - f f_{X\alpha})\right) \lesseqgtr 0.$$

Proof. Applying implicit differentiation to (13), we obtain:

$$\tilde{s}^i_\alpha = -\frac{\tilde{l}^i_\alpha}{\tilde{l}^i_\sigma}$$

Having already established that $\tilde{l}^i_\sigma < 0$, $\text{sgn}\{\tilde{s}^i_\alpha\} = \text{sgn}\{\tilde{l}^i_\alpha\}$. Now,

$$\begin{aligned}
\tilde{l}^i_\alpha &= \sigma^i f_\alpha MRS^i_z - (1-\sigma^i)\frac{f_\alpha}{X} - \sigma_i f_{X\alpha} \\
&= \frac{z^i f_\alpha MRS^i_z}{f} - \frac{\left[(1-\sigma^i)\frac{f}{X} + \sigma^i f_X\right] f_\alpha}{f} + \sigma^i \frac{f_X f_\alpha}{f} - \sigma_i \frac{f_{X\alpha} f}{f} \\
&= \frac{1}{f}\left(f_\alpha(z^i MRS^i_z - MRS^i) + \sigma^i(f_\alpha f_X - f f_{X\alpha})\right).
\end{aligned}$$

□

This condition is slightly more elaborate than the one identified in the previous settings, and given that we allow for $f_X < 0$, we obtain that for individual contest effort to decrease with productivity it is not anymore sufficient for $z^i MRS^i_z > MRS^i$.

Interestingly, for an entire category of production functions, we obtain that a necessary and sufficient condition for determining how individual effort varies with productivity is the same as in our benchmark model. More specifically, for any function admitting an effort-augmenting productivity such that $f(X;\alpha) = \alpha g(X)$, with $g'(X) \gtreqless 0$ and $g''(X) < 0$, we obtain

$$f_\alpha f_X - f f_{X\alpha} = \alpha g g' - \alpha g g' = 0.$$

More generally, how effort varies with the productivity parameter requires conditions that combine properties of the rent generation function with conditions on preferences, where our condition on the direction of change of the ratio of the marginal rate of substitution to the prize share is of fundamental importance.

7 Conclusions

The purpose of this article is to investigate contests allowing for contestants to have more general preferences than have been assumed in existing treatments. We focus on heterogeneous players competing in share contests and prove the existence and uniqueness of a Nash equilibrium. Our main aim is to discover the links between the features of contestants' preferences and the features of the contest equilibrium.

In the conventional literature—where players have (quasi-)linear preferences—aggregate contest efforts are increasing in the prize. We find that when preferences are allowed to be more general than this, and in particular when contestants have diminishing marginal utility over their prize share, this familiar result may no longer hold. We show a key determinant within the contest equilibrium is the rate of change of the marginal rate of substitution between players' share of the prize and their sunk effort. We not only investigate this relationship in a simple Tullock contest, but also in the case of more general contest success functions, and where the prize is endogenously determined.

This encompassing framework now allows the study of contests with much more general preferences and thus increases the applicability of the contest model.

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