1. Introduction

To evaluate \( \int \frac{5}{(x-2)(x+3)} \, dx \) we need to be able to split the fraction into what are called Partial Fractions.

Adding fractions \( \frac{1}{x-2} - \frac{1}{x+3} = \frac{(x+3)-(x-2)}{(x-2)(x+3)} = \frac{5}{(x-2)(x+3)} \)

This implies that in doing the reverse process “Express \( \frac{5}{(x-2)(x+3)} \) in partial fractions” we will get 2 fractions of the form \( \frac{A}{x-2} \) and \( \frac{B}{x+3} \)

Example  Express \( \frac{5}{(x-2)(x+3)} \) in partial fractions

Let \( \frac{5}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} \)

gives \( A(x+3) + B(x-2) = 5 \)

The values of \( x = 2 \) and \( x = -3 \) makes one bracket zero giving the value of one unknown in each case.

Putting \( x = 2 \)  \( A(2+3) + B(2-2) = 5 \)
\( A = 1 \)

Putting \( x = -3 \) \( A(-3+3) + B(-3-2) = 5 \)
\( B = -1 \)

So \( \frac{5}{(x-2)(x+3)} = \frac{1}{x-2} - \frac{1}{x+3} \)

Which is where we started!

Exercise 1

Simplify the following ie express them as single fractions

1. \( \frac{1}{x-1} - \frac{2}{(2x+1)} \)
2. \( \frac{2x}{x^2+1} + \frac{3}{2x-1} \)
3. \( \frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \)
4. \( \frac{3}{(x-1)} + \frac{5}{(x-3)} + 2 \)
2. Some Partial Fractions

(i) Proper fractions

\[ \frac{x^2 + 9x - 25}{(x-1)(2x+1)(x+5)} \]
is a proper fraction as the highest power in the numerator is 2.

\[ \frac{x^3}{(x-1)(2x+1)(x+5)} \]
is improper as the highest power in both numerator and denominator is 3.

\[ \frac{x^4 + 3x^2 - x + 1}{(x-1)(2x+1)(x+5)} \]
is improper as the highest power in numerator (4) is greater than the highest power in the denominator (3).

In the following they are all proper fractions.

Type 1: denominator with only linear factors such as question 1 above

\[ \frac{f(x)}{(x-1)(2x+1)(x+5)} = \frac{A}{x-1} + \frac{B}{2x+1} + \frac{C}{x+5} \]

Type 2: denominator with a quadratic factor such as question 2 above

\[ \frac{f(x)}{(x^2 + 1)(2x+1)(x+5)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{2x+1} + \frac{D}{x+5} \]

Type 3: denominator with repeated linear factor such as question 3 above

\[ \frac{f(x)}{(x-1)^3(2x+1)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{(x+5)} \]

(a) - Type 1

Express \( \frac{3x+1}{(x-1)(2x+1)} \) in partial fractions

Let \( \frac{3x+1}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1} \)

\( 3x+1 \) gives \( A(2x+1) + B(x-1) \)

put \( x = 1 \) \( 3+1 = A(2+1) \quad \Rightarrow A = \frac{1}{3} \)

put \( x = -\frac{1}{2} \) \( 3\left(-\frac{1}{2}\right)+1 = B\left(-\frac{1}{2}-1\right) \quad \Rightarrow B = \frac{1}{3} \)

Solution \( \frac{3x+1}{(x-1)(2x+1)} = \frac{4}{3(x-1)} + \frac{1}{3(2x+1)} \)
(b) - Type 2

Express \( \frac{3x+1}{(x-1)(x^2+1)} \) in partial fractions

Let

\[
\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}
\]

\[
(3x+1) = A(x^2+1) + (Bx+C)(x-1)
\]

putting \( x = 1 \)

\[
3+1 = A(1+1) \Rightarrow A = 2
\]

equate constant term

\[
1 = A - C \Rightarrow C = 1
\]

RHS = \(-B + C = -(2) + 1 = 3 = \text{LHS}\)

Solution

\[
\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x^2+1}
\]

(c) - Type 3

Express \( \frac{36}{(x-1)^2(x+5)} \) in partial fractions

Let

\[
\frac{36}{(x-1)^2(x+5)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+5)}
\]

Multiplying through by \( (x-1)^2(x+5) \)

gives

\[
36 = A(x+5) + B(x+5)(x-1) + C(x-1)^2
\]

putting \( x = 1 \)

\[
36 = A(1+5) \Rightarrow A = 6
\]

putting \( x = -5 \)

\[
36 = C(-1-5)^2 \Rightarrow C = 1
\]

RHS = \( 5A - 5B + C = 30 - 5(-1) + 1 = 36 = \text{LHS}\)

Solution

\[
\frac{36}{(x-1)^2(x+5)} = \frac{6}{(x-1)^2} - \frac{1}{(x-1)} + \frac{1}{(x+5)}
\]
(ii) Improper Fractions

The fraction \( \frac{x^4 - 2x^3 - x^2 - 4x + 4}{(x - 3)(x^2 + 1)} \) is an improper fraction as the highest power of the numerator is 4 and of the denominator is 3.

Before expressing it in partial fractions it is necessary to divide through.

This gives

\[
\frac{x^4 - 2x^3 - x^2 - 4x + 4}{(x - 3)(x^2 + 1)} \equiv x + 1 + \frac{x^2 - 2x + 7}{(x - 3)(x^2 + 1)}
\]

The fraction \( \frac{x^2 - 2x + 7}{(x - 3)(x^2 + 1)} \) can then be put into partial fractions as in example 3 above

\[
giving \quad \frac{x^4 - 2x^3 - x^2 - 4x + 4}{(x - 3)(x^2 + 1)} \equiv x + 1 + \frac{1}{(x - 3)} + \frac{-2}{(x^2 + 1)}
\]

Exercise 2

Express the following in Partial Fractions

\[
\begin{align*}
1. \quad & \frac{3x}{(x + 2)(x - 1)} \\
2. \quad & \frac{2x - 1}{(x + 1)(x - 2)(x + 3)} \\
3. \quad & \frac{2x}{x^2 - 25} \\
4. \quad & \frac{4}{x(x^2 + 4)} \\
5. \quad & \frac{3x^2 + 2x}{(x + 2)(x^2 + 4)} \\
6. \quad & \frac{x^2 + 1}{x(x^2 - 1)} \\
7. \quad & \frac{2}{(x - 1)^2(x + 1)} \\
8. \quad & \frac{x^2 + 3x}{x^2 - 4} \\
9. \quad & \frac{5x - 3}{(x - 2)(x - 3)^2} \\
10. \quad & \frac{x^4 + 1}{x^3 + 2x} \\
11. \quad & \frac{1}{x(x - 1)(x + 1)} \\
12. \quad & \frac{16}{(x - 1)^2(x + 1)^3}
\end{align*}
\]
3. Integration

**Note** It is important to recognise certain standard integrals and methods here.

1. The use of \( \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c \)

2. Using constant multipliers to simplify integration such as
\[
\int \frac{x}{2x^2 + 3} \, dx = \frac{1}{4} \int \frac{4x}{2x^2 + 3} \, dx = \frac{1}{4} \ln(2x^2 + 3) + c
\]

3. Splitting up such expressions as \( \frac{2x + 1}{x^2 + 1} \) to get \( \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} \)
giving \( \int \frac{2x + 1}{x^2 + 1} \, dx = \int \frac{2x}{x^2 + 1} \, dx + \frac{1}{x^2 + 1} \, dx = \ln(x^2 + 1) + \tan^{-1} x + c \)

**Examples**

(a) Simplify \( \int \frac{3x + 1}{(x - 1)(2x + 1)} \, dx \)

From example. (a) on page 1
\[
\int \frac{3x + 1}{(x - 1)(2x + 1)} \, dx = \frac{4}{3} \int \frac{1}{x-1} \, dx + \frac{1}{3} \int \frac{2}{2x+1} \, dx
\]
\[
= \frac{4}{3} \ln|x-1| + \frac{1}{6} \ln|2x+1| + c
\]

(b) Simplify \( \int \frac{3x + 1}{(x - 1)(x^2 + 1)} \, dx \)

From example. (b) on page 2
\[
\int \frac{3x + 1}{(x - 1)(x^2 + 1)} \, dx = \int \frac{2}{x-1} + \frac{1-2x}{x^2 + 1} \, dx
\]
\[
= \ln(x-1) + \tan^{-1} x - \ln(x^2 + 1) + c
\]
(c) Find the value of $\int_{2}^{4} \frac{36}{2(x-1)^2(x+5)} \, dx$

From example (c) on page 2 we can write

$$\int_{2}^{4} \frac{36}{(x-1)^2(x+5)} \, dx = \int_{2}^{4} \left( \frac{6}{2(x-1)^2} - \frac{1}{(x-1)} + \frac{1}{(x+5)} \right) \, dx$$

$$= \left[ \frac{-6}{x-1} - \ln|x-1| + \ln|x+5| \right]_{2}^{4}$$

$$= \left[ \frac{-6}{4} - \ln3 + \ln9 \right] - \left[ \frac{-6}{1} - \ln1 + \ln7 \right]$$

$$= -2 - \ln3 + \ln9 + 6 - 0 - \ln7$$

$$= 4 + \ln\left( \frac{9 \times 7}{3} \right) = 4 + \ln\left( \frac{3}{7} \right)$$

(d) Evaluate $\int_{4}^{5} \frac{24x^3(x-3)}{(x-1)(2x+1)} \, dx$

Multiplying out top and bottom and dividing gives:

$$\frac{24x^3(x-3)}{(x-1)(2x+1)} = \frac{24x^4 - 72x^3}{2x^2 - x - 1}$$

$$= 12x^2 - 30x - 9 - \frac{39x + 9}{2x^2 - x - 1}$$

Expressing $\frac{39x + 9}{2x^2 - x - 1}$ in partial fractions

Let $\frac{39x + 9}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$

then $39x + 9 = A(2x+1) + B(x-1)$

putting $x = 1$ gives $48 = 3A \Rightarrow A = 16$

putting $x = 0$ gives $9 = A - B \Rightarrow B = 7$

This gives $\frac{24x^3(x-3)}{(x-1)(2x+1)} = 12x^2 - 30x - 9 - \frac{16}{(x-1)} - \frac{7}{(2x+1)}$

Hence $\int_{4}^{5} \frac{24x^3(x-3)}{(x-1)(2x+1)} \, dx = \int_{4}^{5} \left( 12x^2 - 30x - 9 - \frac{16}{(x-1)} - \frac{7}{(2x+1)} \right) \, dx$

$$= \left[ 4x^3 - 15x^2 - 9x - 16\ln|x-1| - \frac{7}{2}\ln|2x+1| \right]_{4}^{5}$$
\[
\begin{align*}
&= \left[500 - 375 - 45 - 16\ln 4 - \frac{7}{2}\ln 11\right] - \left[256 - 240 - 36 - 16\ln 3 - \frac{7}{2}\ln 9\right] \\
&= 80 - 32\ln 2 - \frac{7}{2}\ln 11 - \left[- 20 - 16\ln 3 - 7\ln 3\right] \\
&= 80 - 32\ln 2 - \frac{7}{2}\ln 11 + 20 + 16\ln 3 + 7\ln 3 \\
&= 100 - 32\ln 2 + 23\ln 3 - \frac{7}{2}\ln 11 \\
\end{align*}
\]

(e) Evaluate \( \int_1^2 \frac{1+x}{x^2(x^2+1)} \, dx \)

First express \( \frac{1+x}{x^2(x^2+1)} \) in Partial Fractions

Writing \( \frac{1+x}{x^2(x^2+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1} \)

and equating terms gives \( A = 1, \ B = 1, \ C = -1, \ D = -1 \)

Hence

\[
\int_1^2 \frac{1+x}{x^2(x^2+1)} \, dx = \int_1^2 \frac{1}{x^2} + \frac{1}{x} - \frac{x+1}{x^2+1} \, dx
\]

\[
= \int_1^2 x^{-2} + \frac{1}{x} - \frac{1}{2} \left( \frac{2x}{x^2+1} \right) - \frac{1}{x^2+1} \, dx
\]

\[
= \left[ -x^{-1} + \ln x - \frac{1}{2} \ln \left( x^2+1 \right) - \tan^{-1} x \right]_1^2
\]

\[
= \left( -\frac{1}{2} + \ln 2 - \frac{1}{2} \ln 5 - \tan^{-1} 2 \right) - \left( -1 + \ln 1 - \frac{1}{2} \ln 2 - \tan^{-1} 1 \right)
\]

This can be simplified depending on what is required.

**Exercise 3**

Find the following integrals (You will find your solutions to the previous exercise will help in some cases)

1. \( \int_2^5 \frac{3x}{(x+2)(x-1)} \, dx \)
2. \( \int \frac{10(2x-1)}{(x+1)(x-2)(x+3)} \, dx \)
3. \( \int \frac{2x}{x^2-25} \, dx \)
4. \( \int \frac{4}{x(x^2+4)} \, dx \)
5. \( \int_0^2 \frac{3x^2+2x}{(x+2)(x^2+4)} \, dx \)
6. \( \int \frac{x^2+1}{x(x^2-1)} \, dx \)
7. \( \int \frac{4}{(x-1)^2(x+1)} \, dx \)
8. \( \int_6^8 \frac{2(x^2+3x)}{x^2-4} \, dx \)
9. \( \int \frac{5x-3}{(x-2)(x-3)^2} \, dx \)
10. \( \int_1^2 \frac{x^4+1}{x^3+2x} \, dx \)
Exercise 1

1. \[ \frac{3}{(x-1)(2x+1)} \]
2. \[ \frac{7x^2 - 2x + 3}{(x^2 + 1)(2x-1)} \]
3. \[ \frac{x^2 - x + 1}{(x-1)^3} \]
4. \[ \frac{2(x^2 - 4)}{(x-1)(x-3)} \]

Exercise 2

1. \[ \frac{2}{x+2} + \frac{1}{x-1} \]
2. \[ \frac{1}{2(x+1)} + \frac{1}{5(x-2)} - \frac{7}{10(x+3)} \]
3. \[ \frac{1}{x-5} + \frac{1}{x+5} \]
4. \[ \frac{1}{x} - \frac{x}{x^2 + 4} \]
5. \[ \frac{1}{x+2} + \frac{2x-2}{x^2 + 4} \]
6. \[ \frac{1}{x+1} + \frac{1}{x-1} - \frac{1}{x} \]
7. \[ \frac{1}{(x-1)^2} - \frac{1}{2(x-1)} + \frac{1}{2(x+1)} \]
8. \[ \frac{1+5}{2(x-2)} + \frac{1}{2(x+2)} \]
9. \[ \frac{12}{(x-3)^2} - \frac{7}{x-3} + \frac{7}{x-2} \]
10. \[ x + \frac{1}{2x} - \frac{5x}{2(x^2 + 2)} \]
11. \[ \frac{1}{2(x-1)} + \frac{1}{2(x+1)} - \frac{1}{x} \]
12. \[ \frac{2}{(x-1)^2} - \frac{3}{x-1} + \frac{4}{(x+1)^3} + \frac{4}{(x+1)^2} + \frac{3}{x+1} \]

Exercise 3

Note: most of the partial fractions were worked out in the previous exercise. The ‘best’ form of the answer depends on what comes next!

1. \[ 2 \ln \frac{7}{2} \]
2. \[ 5 \ln |x+1| + 2 \ln |x-2| - 7 \ln |x+3| + c = \ln \left| \frac{(x+1)^5 (x-2)^2}{(x+3)^7} \right| + c \]
3. \[ \ln |x-5| + \ln |x+5| + c = \ln |x^2 - 25| + c \]
4. \[ \frac{1}{2} \ln \frac{5}{2} \]
5. \[ 2 \ln 2 - \frac{\pi}{4} \]
6. \[ \ln |x+1| + \ln |x-1| - \ln |x| + c = \ln \left| \frac{x^2 - 1}{x} \right| + c \]
7. \[ \ln |x+1| - \ln |x-1| - \frac{2}{x-1} + c = \ln \left| \frac{x+1}{x-1} \right| - \frac{2}{x-1} + c \]
8. \[ \ln 5 + 5 \ln 3 - 7 \ln 2 + 4 = \ln \left( \frac{5 \times 3^5}{2^7} \right) + 4 = \ln \left( \frac{1215}{128} \right) + 4 \]
9. \[ 7 \ln |x-2| - 7 \ln |x+3| - \frac{12}{x-3} + c = 7 \ln \left| \frac{x-2}{x+3} \right| - \frac{12}{x-3} + c \]
10. \[ \frac{3(2 - \ln 2)}{4} \]

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Many thanks for use of these materials. Any comments can be sent to the email above or to mathsskills@strath.ac.uk